SECONDARY MATHEMATICS
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SECONDARY MATHEMATICS

I.

By

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Introduction

THE growth of this series of Mathematics for Secondary Schools, has covered a period of seven years, and has been simultaneous with the growth and development of the shop, laboratory, and drawing courses in Cass Technical High day school, as well as in the evening and continuation classes.

The authors have had clearly in mind the necessity of first developing a sequence of mathematics that would enable the student to recognize fundamental principles and apply them in the shop, drawing room, and laboratory; and, second to so develop the course that each year's work would be a unit and not depend upon subsequent development for intelligent application.

It has been assumed that the school work-shop, drawing room, and laboratory would furnish opportunity to apply mathematics and that it was not necessary to exhaust every possible application in the mathematics class.

The authors have been aware of the popular demand for a closer union of algebra and geometry, but have recognized that demand only when the union came about naturally and would assist the mathematical sequence desired.

Instructors in the wood shop, pattern shops, machine shop, drawing rooms, chemistry, physics, and electrical laboratories, etc., have furnished examples of mathematical application incident to the respective subjects. Hundreds of problems arising in the industries, have been brought in by the machinists, sheet metal workers, carpenters, electrical workers, pattern makers, draughtsmen, etc., etc., coming to the evening and continuation classes. Complete charts of machine shop work and electrical distribution requirements have been made, including a statement of the required sequence of mathematics. All of this material has been classified, with a view to the mathematical sequence.

The net result is a series of Mathematics so organized that a mastery of the text makes it possible for a student to use mathematics intelligently in the various departments of the school, in the industries, and at the same time prepare for college mathematics.

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CHAPTER I

THE EQUATION

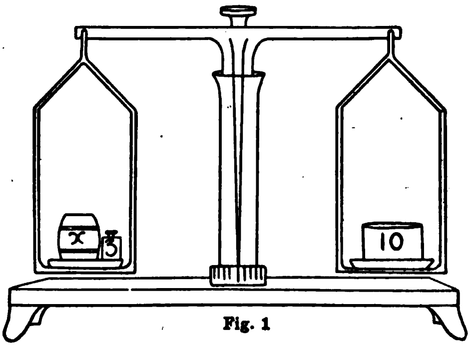


Fig. 1

1 In order to find the weight of an object, it was placed on one pan of perfectly balanced scales (Fig. 1). It, together with a 3-lb. weight, balanced a 10-lb. weight on the other pan. If 3 lbs. could be taken from each pan, the object would be balanced by 7 lbs. This may be expressed by the *equation*, $x+3=10$, in which the expressions $x+3$ and 10 denote the weights in the pans, the sign (=) of equality denotes the perfect balance of the scales, and x is to be found.

2 Equation: An *equation* is a statement that two expressions are equal. The two expressions are the *members* of the equation, the one at the left of the equality sign being called the *first member*, and the one at the right, the *second member*.

3 From the explanatory problem, it will be seen that *the same number may be subtracted from both members of an equation.*

Oral Problems:

Solve for x:

1. $x+7=21$
2. $x+2=3$
3. $x+1.1=3.5$
4. $x+2\frac{5}{6}=7\frac{1}{2}$
5. $x+\frac{5}{7}=\frac{11}{12}$

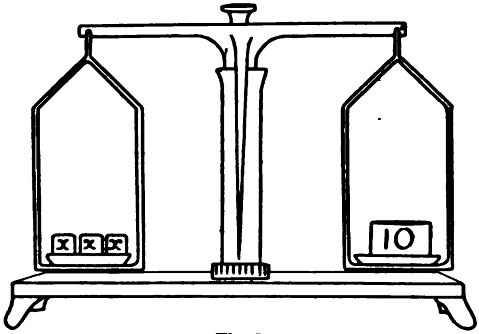


Fig. 2

4 It is required to find the weight of a casting. It is found that 3 of them exactly balance a 10-lb. weight (Fig. 2). If the weight in each pan could be divided by 3, one casting would be balanced by $3\frac{1}{3}$ lbs. This may be expressed by the equation,

$$3x = 10,$$

$$x = 3\tfrac{1}{3}, \text{ (dividing both members by 3)}$$

5 From this explanatory problem, it will be seen that *both members of an equation may be divided by the same number.*

Oral Problems:

Solve for x:

1. $4x = 12$
2. $2x = 16$
3. $5x = 9$
4. $11x = 33$
5. $1.1x = 12.1$

Example: Solve for x: $5x + 12 = 37$

$5x = 25$ Why?

$x = 5$ Why?

Exercise 1

Solve for the unknown:

1. $x+1=5$
2. $x+7=9$
3. $2a+6=16$
4. $3x+7=28$
5. $5s+17=62$
6. $9x+12=93$
7. $2x+1=6$
8. $5y+3=15$
9. $4n+3.2=15.2$
10. $12m+8=98$
11. $9x+8=116$
12. $7w+5\frac{2}{3}=12\frac{2}{3}$
13. $28t+14=158$
14. $3x+4\frac{1}{2}=9$
15. $15s+.5=26$
16. $11x+\frac{1}{4}=\frac{89}{4}$
17. $1.2x+2=14$
18. $4.6x+8=100$
19. $6.3x+2.4=15$
20. $7.1m+.55=9.07$

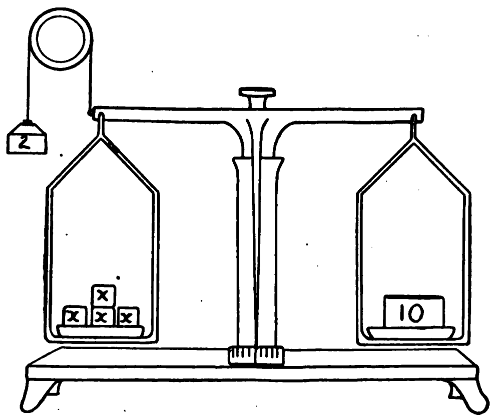


Fig. 3

6 If an apparatus is arranged as in Fig. 3, it is seen that if the upward pull of 2 lbs. be removed, 2 lbs. would have to be

put upon the other pan to keep the scales balanced. This may be expressed by the equation,

$$4x-2=10$$
$$4x=12 \quad \text{(Adding 2 to both members)}$$
$$x=3 \quad \text{(Dividing both members by 4)}$$

7 From this problem, it will be seen that *the same number may be added to both members of an equation.*

Oral Problems:

Solve for x:

1. $3x-4=8$
2. $7x-1=15$
3. $4x-\frac{3}{4}=7\frac{1}{4}$
4. $5x-.1=.9$
5. $2x-\frac{1}{3}=6\frac{1}{3}$

Exercise 2

Solve for the unknown:

1. $x-7=10$
2. $2x-13=11$
3. $5x-17=13$
4. $4x-11=25$
5. $3x-7=15$
6. $12x-4=44$
7. $7m-5=31$
8. $4x-18=18$
9. $17t-3\frac{1}{5}=13\frac{4}{5}$
10. $11x-9=90$
11. $13r-21=44$
12. $12s-35=41$
13. $7f-4=26$
14. $4x-3=16$
15. $9x-3.2=14.8$
16. $3m-2=3.1$
17. $14x-5=21$
18. $2.1x-3.2=3.1$
19. $.5y-4=5.5$
20. $3x-9\frac{1}{2}=8.5$

Exercise 3. Review

Solve for the unknown:

1. $9x-8=46$
2. $8x-7=53$
3. $5x+7=28$
4. $28m-9=251$
5. $16y+13=73$
6. $3w-1\frac{1}{3}=1\frac{2}{3}$
7. $19t-.2=3.6$
8. $6.37n+3.92=73.99$
9. $.4x+.02=.076$
10. $2s+2\frac{1}{4}=9\frac{3}{4}$
11. Two times a number increased by 43 equals 63. Find the number.
12. If 10 be added to 3 times a number, the result is 50. What is the number?
13. Five times a number decreased by 6 equals 39. Find the number.
14. If 55 be subtracted from 7 times a number, the result is 22. What is the number?
15. If to 57 I add twice a certain number, the result is 171. What is the number?
16. State the first five problems in this exercise in words.

8 How many yards of cloth are 7 yds. and 5 yds.?

How many dozens of eggs are 12 doz. and 3 doz.?

How many bushels of wheat are 8 bushels and 11 bushels?

How many b's are 4 b's and $7\frac{1}{2}$ b's?

How many x's are 3x and 9x?

In such expressions as $2a+3x+4+2x+7+3a$, 2a and 3a may be combined, 3x and 2x, and also 4 and 7, making the expression equal to $5a+5x+11$. 2a and 3a, 3x and 2x, 4 and 7 are called *similar terms*.

Example 1. Solve for x:

$$4x+13x-7x=40$$
$$10x=40 \quad \text{(combining similar terms)}$$
$$x=4. \quad \text{Why?}$$

Example 2. Solve for x:

$$14x+7-2x=43$$
$$12x+7=43 \quad \text{Why?}$$
$$12x=36 \quad \text{Why?}$$
$$x=3 \quad \text{Why?}$$

9 $8-7+3=?$ $\qquad$ $8x-7x+3x=?$

$8+3-7=?$ Similarly $8x+3x-7x=?$

$3-7+8=?$ $\qquad$ $3x-7x+8x=?$

10 These problems illustrate the principle that *the value of an expression is unchanged if the order of its terms is changed, provided each term carries with it the sign at its left.*

NOTE: If no sign is expressed at the left of the first term, the sign (+) is understood.

Example 1. $15-3x+11x=39$

$$8x+15=39 \quad \text{Why?}$$
$$8x=24 \quad \text{Why?}$$
$$x=3 \quad \text{Why?}$$

Example 2: $11y-4+21=50$

$$11y+17=50 \quad \text{Why?}$$
$$11y=33 \quad \text{Why?}$$
$$y=3 \quad \text{Why?}$$

Exercise 4

Solve:

1. $4x-x=12$
2. $11x+3x=35$
3. $14x-3x=44$
4. $3x+7x=90$
5. $9y-9y+8y=40$
6. $4s+3s-2s=17$
7. $3.2x+2.3x=110$
8. $1.3y-2.7y+3.3y=57$
9. $11.2x+7.8x=57$
10. $1.1s-1.4s+11s=26.75$

Exercise 5

Solve:

1. $x-18=17$
2. $x+18=21$
3. $2y-16=30$
4. $3m-m=21$
5. $3m-1=23$
6. $6.5x-1.1=50.9$
7. $4x+3x-3=25$
8. $11y-4y-7=28$
9. $12x-8x+6+3x=8+12$
10. $25x+20-7x-5+5x=56+5$
11. $8x+60+4x-50+3x-7x=20$
12. $2-2x+7x=42.5$
13. $3y+1.2+2y=46$
14. $x-1.25x+12.7+3.5x=38.7$
15. $2x+15.8-2.3x+14.5x=186.2$
16. $6.15y-1.65y+7.8=57.3$
17. $8y+6.875+2y=46.875$
18. $z-8.73+5.37z=61.34$
19. $5t-8.75t+6.87+8t=57.87$
20. $3.73x-9.23+15x=65.69$

11 Equations often arise in which the unknown appears in both members. In that case, aim to make the term containing the *unknown* disappear from *one* member, and the one containing the *known*, from the *other* member.

Example 1: $3x-1=x+3$

$3x=x+4$ (adding 1 to both members).

$2x=4$ (subtracting x from both members).

$x=2$ Why?

Note that in adding or subtracting a term from both members, it must be combined with a similar term.

Example 2:

$5x+4-3x-1=7-x+2$

$2x+3=9-x$ (combining similar terms in each member).

$2x=6-x$ Why?

$3x=6$ Why?

$x=2$ Why?

Exercise 6

Solve:

1. $2x-6=x$
2. $2x+3=x+5$
3. $13x-40=8+x$
4. $7y-7=3y+21$
5. $9x-8=25-2x$
6. $20+10x=38+4x$
7. $3x+9+2x+6=18+4x$
8. $5x+3-x=x+18$
9. $7m-18+3m=12+2m+2$
10. $18+6m+30+6m=4m+8+12+3m+3+m+29$

11. $25x+20-7x-5=56-5x+5$

12. $10x-61-12x+27x=8x-41+20+4x+25$

13. $25\frac{2}{3}+5x+6x+9\frac{1}{2}-2x=180-8x-8\frac{5}{6}$

14. $2.8x+39.33+x=180-1.2x+32.09-7.16$

15. $5x+26\frac{2}{5}+9x=360-5x-143\frac{3}{5}$

12 If an object in one pan of scales will balance a 4-lb. weight in the other, it will be readily seen that 5 objects of the same kind would need 20 lbs. to balance them. This may be expressed by the equation, $x=4$

$5x=20$ (multiplying both members by 5).

13 From this problem, it will be seen that *both members of an equation may be multiplied by the same number.*

This principle is needed when the equation contains fractions. The process of making fractions disappear from an equation is called *clearing of fractions.*

14 **RULE: To clear an equation of fractions, multiply both members by the lowest common denominator (L. C. D.) of all the fractions contained in the equation.**

Example 1: $\frac{x}{2}+3=4$

$x+6=8$ (multiplying both members by 2).

$x=2$ Why?

Example 2: $\frac{r}{3}-\frac{r}{7}=\frac{16}{3}$

$7r-3r=112$ (multiplying both members by 21).

$4r=112$ Why?

$r=28$ Why?

Example 3: $\frac{m}{4}-3\frac{3}{10}+\frac{m}{5}=\frac{7}{5}-\frac{m}{3}$

$$15m-198+12m=84-20m \quad \text{Why?}$$
$$27m-198=84-20m \quad \text{Why?}$$
$$47m-198=84 \quad \text{Why?}$$
$$47m=282 \quad \text{Why?}$$
$$m=6 \quad \text{Why?}$$

15 The four principles used thus far may be more generally stated as follows:

1. *If equals are added to equals, the results are equal.*
2. *If equals are subtracted from equals, the results are equal.*
3. *If equals are multiplied by equals, the results are equal.*
4. *If equals are divided by equals, the results are equal.*

Exercise 7

Solve:

1. $\frac{m}{3}-\frac{m}{6}=10$
2. $\frac{x}{5}+\frac{x}{4}=9$
3. $\frac{2r}{3}+\frac{3r}{4}=17$
4. $x+\frac{1}{2}x=6$
5. $x-\frac{1}{2}x=7$
6. $\frac{x}{2}-\frac{2}{3}=\frac{x}{6}$
7. $\frac{y}{3}=\frac{y}{7}+16$
8. $\frac{2}{5}x+3=\frac{x}{3}+4$
9. $\frac{2x}{9}+\frac{x}{6}=\frac{x}{18}+\frac{1}{3}$
10. $\frac{3x}{7}-\frac{1}{3}=\frac{x}{21}+5$

11. $1\frac{1}{3}s+\frac{2}{7}s=s+13$

12. $\frac{5}{6}x+2\frac{5}{12}=3+\frac{x}{4}$

13. $\frac{r}{4}+4r-\frac{5r}{3}=26+1\frac{1}{2}r$

14. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}+\frac{x}{10}=82$

15. $7x+\frac{4x}{7}+\frac{x3}{5}+23=\frac{4x}{10}+5\frac{1}{5}x+113$

16 Sometimes it is convenient to make the term containing the *unknown* disappear from the *first* member. and the one containing the *known*, from the *second*.

Example 1: $x+6=3x-2$

$6=2x-2$ Why?

$8=2x$ Why?

$x=4$ Why?

Example 2: $\frac{16}{3x}=4$

$16=12x$ Why? (L. C. D. is 3x)

$x=1\frac{1}{3}$ Why?

Exercise 8

Solve:

1. $\frac{15}{a}=5$

2. $\frac{5}{a}=15$

3. $\frac{3}{4x}=2$

4. $\frac{3x}{4}=\frac{7}{2}$

5. $\frac{16}{5}=\frac{2x}{3}$

6. $14=x+9$

7. $17=2x-3$

8 $x+10=2x-9$

9. $2x-2\frac{1}{4}=5x-17\frac{1}{4}$

10. $7x+20-3x=60+4x-50+8x$

11. $3m+60=15m+3-2m+7$

12. $\frac{3n}{2}+47=\frac{6n}{7}+4n$

13. $\frac{5a}{2}-1=\frac{17a}{3}-\frac{5}{3}-2\frac{1}{2}a$

14. $.1x+6.2=.3x+.2$

15. $10+.1x=5+\frac{1}{5}x$

Exercise 9

Solve:

1. $\frac{7m}{6}-8=5\frac{1}{2}-\frac{13m}{12}$

2. $7x-8=6x+\frac{1}{5}x$

3. $\frac{2x}{5}-\frac{x}{6}=25\frac{1}{2}-\frac{x}{3}$

4. $\frac{2x}{3}+3=\frac{5}{6}x-2$

5. $\frac{2t}{3}+\frac{5t}{9}=\frac{t}{6}+\frac{t}{2}+10$

6. $\frac{1}{2}+\frac{x}{5}-\frac{x}{6}=\frac{x}{3}-\frac{x}{4}$

7. $\frac{y}{12}-\frac{3}{8}+2\frac{3}{4}=\frac{5}{4}-\frac{y}{8}+\frac{y}{4}$

8. $\frac{5}{3}x-\frac{4}{5}x+4\frac{2}{5}=3x+\frac{z}{15}$

9. $11\frac{7}{12}x-1\frac{1}{6}x-302=60+1\frac{1}{3}x+183$

10. $x-3\frac{2}{3}+\frac{1}{2}x=9\frac{1}{2}-\frac{x}{3}$

Exercise 10

1. Five times a certain number equals 155. What is the number?

2. Four times a number increased by 7 equals 43. Find the number.

3. Twelve times a number decreased by .18 is equal to 17.82. Find the number.

4. There are three numbers whose sum is 72. The second number is three times the first, and the third is four times the first. What are the numbers?

5. The sum of two numbers is 12 and the first is 4 more than the second. What are the numbers?

6. If 10 is subtracted from three times a number, the result is twice the number. Find the number.

7. If $\frac{4}{5}$ of a number is increased by 6, the result is 30. Find the number.

8. The sum of $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ of a number is 26. What is the number?

9. Divide 19 into two parts so that one part is 5 more than the other.

10. Divide 19 into two parts so that one part is 5 times the other.

11. Divide \$24 between two persons so that one shall receive \$$2\frac{1}{2}$ more than the other.

12. A farmer has 4 times as many sheep as his neighbor. After selling 14, he has $3\frac{1}{3}$ times as many. How many had each before the sale?

13. Two men divide \$2123 between them so that one receives \$8 more than 4 times as much as the other. How much does each receive?

14. Three candidates received in all 1020 votes. The first received 143 more than the third, and the second 49 more than the third. How many votes did each receive?

15. A man spent a certain sum of money for rent, $\frac{2}{3}$ as much for groceries, \$2 more for coal than for rent, and \$28 for incidentals. In all he paid out \$100.00. How much did he spend for each?

16. A farmer has 24 acres more than one neighbor and 62 acres less than another. The three together own one square mile of land. How much has each?

17. A man traveled a certain number of miles on Monday, $\frac{5}{7}$ as many on Tuesday, $\frac{2}{3}$ as many on Wednesday as on Monday, and on Thursday 10 miles less than twice as many as he did on Monday. How far did he travel each day if his trip covered 82 miles?

18. One man has 3 times as many acres of land as another. After the first sold 60 acres to the second, he had 40 acres more than the second then had. How many acres did each have before the transaction?

19. One boy has \$10.40 and his brother has \$64.80. The first saves 20 cents each day, and his brother spends 20 cents each day. In how many days will they have the same amount?

20. A man after buying 27 sheep finds that he has $1\frac{3}{7}$ times his original flock. How many sheep had he at first?

CHAPTER II

EVALUATION

17 Definite Numbers: The numerals used in arithmetic have *definite* meanings. For example, the numeral 7 is used to represent a definite thing. It may be 7 yards, 7 pounds, 7 cubic feet or 7 of any other unit. Also in finding the circumference of a circle, we multiply the diameter by π which has a *fixed value.* Numerals and letters which represent *fixed values* are called *definite numbers.*

18 General Numbers: The area of a rectangle is found by multiplying the base by the altitude. This may be expressed by $b \times a$, in which the value of *b* may be 10 ft., 6 in., 30 rds., or *any* number of any unit used to measure length, and *a* may be any number of a like unit. Letters which may represent *different* values in different problems are called *general numbers.*

19 Signs: When the multiplication of two or more factors is to be indicated, the sign of multiplication is often omitted or expressed by the sign $(\cdot)$. Thus $7 \times a \times b \times m$ is written $7 \cdot a \cdot b \cdot m$ or more often 7abm.

NOTE: Care should be taken in the use of the sign $(\cdot)$ to distinguish it from the decimal point. $7 \cdot 9$ means 7×9, 7.9 means $7\frac{9}{10}$.

20 Coefficient: The expression 7abm may be thought of as $7ab \cdot m$, $7\ a \cdot bm$, $7 \cdot abm$, or $7b \cdot am$, etc. 7ab, 7a, 7, and 7b are called the *coefficients* of m, bm, abm, and am respectively.

1. In the following, what are the coefficients of x? 4abx; $\frac{4}{3}$xyz; 17mxw.

2. Name the coefficients of ab in the following: $3\frac{1}{3}$axby; $\frac{5}{2}$mabz; .9bnsa.

3. What is the coefficient of 17 in 17mxw?

The *coefficient* of a factor or of the product of any number of factors, is the product of all the *remaining* factors. In 8axy, 8 is the *numerical* coefficient. The numerical coefficient 1 is *never* written. 1axy is written axy.

21 Power: If all the factors in a product are the same, as $x \cdot x \cdot x \cdot x$, the product is called a *power*. $x \cdot x \cdot x \cdot x$ is read "x fourth power" and is written x^4. $a \cdot a \cdot a \cdot a \cdot a$ is read "a fifth power" and is written a^5. $b \cdot b$ or b^2 is "b second power" but is more often read "b square." In the same way $b \cdot b \cdot b$ (b^3) is called "b-cube."

22 Exponent: The small number written at the right and above a number is called its *exponent* and it indicates the power of the number. The exponent 1 is never written. x means x^1 or "x first power."

23 Base: The number to be raised to a power is called the *base.*

Name the numerical coefficients, bases and exponents in the following:

$\frac{1}{3}{}^{9}x^7$, $5\frac{1}{2}a^{10}$, $3 \cdot 7m^2n^4$, $\frac{4}{3}\pi r^3$, $1\frac{5}{6}m$ $1\frac{5}{6}m^x$.

24 Sign of Grouping: The *Sign of Grouping* most commonly used is the *parenthesis* ( ) and means that the parts enclosed are to be taken as a single quantity. For example, 3(x-y) means that x-y is to be multiplied by 3 making 3x - 3y. $(x - y)^3$ means (x-y)(x-y)(x-y).

25 Evaluation: Evaluation of an expression is the process of finding its *value* by substituting *definite* numbers for *general* numbers in the expression, and performing the operations indicated.

Example 1: Evaluate $4a^2x^3$ if $a=3$, $x=2$.

$4a^2x^3=4\cdot3^2\cdot2^3=4\cdot9\cdot8=288.$

Example 2: Find the value of $\frac{a^2}{m^3}-\frac{5b^4}{c^2}+\frac{m^5}{2a^3}$

when $a=1$, $b=2$, $c=5$, $m=2$.

$$\frac{a^2}{m^3}-\frac{5b^4}{c^2}+\frac{m^5}{2a^3}=\frac{1^2}{2^3}-\frac{5\cdot2^4}{5^2}+\frac{2^5}{2\cdot1^3}$$

$$=\frac{1}{8}-\frac{80}{25}+\frac{32}{2}$$

$$=\frac{1}{8}-\frac{16}{5}+16$$

$$=\frac{5-128+640}{40}$$

$$=\frac{517}{40}=12\tfrac{37}{40}$$

Example 3: Evaluate $a(a-b+y^2)$ when $a=13$, $b=3$, $y=4$.

$$a(a-b+y^2)=13(13-3+4^2)$$
$$=13\cdot26$$
$$=338$$

Exercise 11

Evaluate the following if $a=8$, $b=6$, $c=4$, $d=2$, $x=9$:

1. $2x$
2. x^2
3. $3x$
4. x^3
5. $4x$
6. x^4
7. $3x^2$
8. $(3x)^2$
9. $11ax$
10. $2abcd$
11. $2a^2x^3$
12. x^2-a^2

13. $x(a+b)$
14. $4b(x-c)$
15. $a^2+2ab+b^2$
16. $c^2-2cd+d^2$
17. $\frac{1}{x}-\frac{1}{b}+\frac{1}{d}$
18. $(x+a)(c-d)$
19. $\sqrt{4ad}$
20. $ab(c-3)$

Exercise 12

Find the value of the following, when $a=2$, $b=3$, $c=7$, $d=4$, $m=1$, $x=5$:

1. $\frac{1}{2}a^3x^2c$
2. x^3-a^3
3. x^3+d^3
4. $3b^2-4m^2$
5. x^3d-a^2m
6. $2a^2x^3(c-d)$
7. $\frac{b}{a^2}+\frac{m}{d}$
8. $\frac{1}{2}(x+a)c$
9. $\frac{1}{2}a^3x^2c(b^3-d^2)$
10. $\frac{a^2}{b^2}+\frac{c^2}{d^2}-\frac{x^2}{a^3}$
11. $(3x+7)(c-2)$
12. $\sqrt{b^2+d^2}$
13. $\frac{3a^2}{bd}(x^2-c^2+25)$
14. $a^3(x-c+3m)(c^2+d^2)$
15. $\sqrt[3]{ad}$
16. $\frac{ab}{2d}(x^2+a^2-b^2)(c^2-d^2-m^2)$
17. $\sqrt{x(a+b)}$
18. $\sqrt[3]{d(a+b)+c}$
19. $\sqrt[3]{5x(a+b)}$
20. $(a+b)(b+c)-(b+c)(x+d)+(x+d)(d+m)$

Evaluation of Formulas

26 A *Formula* is the statement of a rule or principle in terms of *general numbers*. For example, *distance* traveled is *equal* to *rate* times *time*.

Formula, $d=r \cdot t$

Example 1: Evaluate $b=\frac{lwt}{12}$ (Formula for board feet)

when $l=16'$, $w=8''$, $t=2''$

$$b=\frac{16 \cdot 8 \cdot 2}{12}=21\tfrac{1}{3}$$

Example 2: Evaluate $A=\frac{1}{2}h(b+b')$ (Area of a trapezoid)

if $h=3\frac{3}{16}$, $b=12\frac{1}{4}$, $b'=6\frac{1}{2}$

$$A=\tfrac{1}{2}\cdot 3\tfrac{3}{16}\ (12\tfrac{1}{4}+6\tfrac{1}{2})$$

$$A=\tfrac{1}{2}\cdot 3\tfrac{3}{16}\cdot 18\tfrac{3}{4}$$

$$A=\frac{1}{2}\cdot\frac{51}{16}\cdot\frac{75}{4}$$

$$=\frac{3825}{128}=29\tfrac{113}{128} \text{ or } 29.102-$$

Perimeter Formulas

27 The *perimeter* of a figure enclosed by straight lines is the sum of its sides.

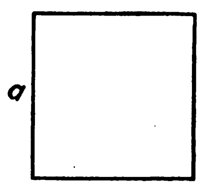


Fig. 4. Square

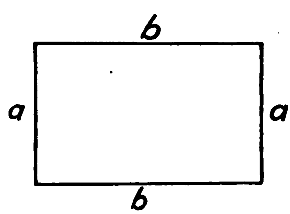


Fig. 5. Rectangle

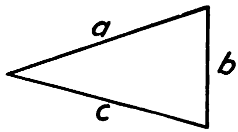


Fig. 6. Triangle

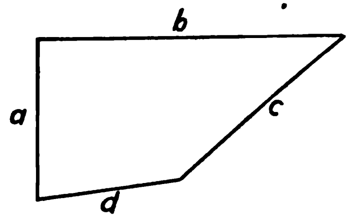


Fig. 7. Quadrilateral

Exercise 13

1. The perimeter of a square (Fig. 4) is equal to 4 times one side. $P=4a$. Find P, if $a=9$.

2. Find the value of P, in $P=4a$, if $a=1\frac{1}{2}$.

3. Find the value of P, in $P=4a$, if $a=1.175$.

4. The perimeter of a rectangle (Fig. 5) is equal to $a+b+a+b=2a+2b=2(a+b)$. $P=2(a+b)$. Find P, if $a=3$, $b=5$.

5. Find P, in $P=2(a+b)$, if $a=\frac{1}{6}$, $b=\frac{2}{3}$.

6. Find P, in $P=2(a+b)$, if $a=1.7862$, $b=2.1324$.

7. The perimeter of a triangle (Fig. 6) is expressed by the formula, $P=a+b+c$. Find P, if $a=7$, $b=11$, $c=19$.

8. Evaluate $P=a+b+c$, if $a=\frac{5}{6}$, $b=\frac{3}{7}$, $c=\frac{2}{3}$.

9. Find the value of P, in $P=a+b+c$, if $a=7.621$, $b=8.37$, $c=1.3$.

10. The perimeter of a quadrilateral (Fig. 7) is expressed by the formula, $P=a+b+c+d$. Find P, if $a=20$, $b=15$, $c=13$, $d=14$.

11. Evaluate $P=a+b+c+d$, when $a=1\frac{3}{4}$, $b=1\frac{5}{6}$, $c=1\frac{7}{12}$, $d=1\frac{1}{3}$.

12. Find P, in $P=a+b+c+d$, if $a=172.32$, $b=96.3$, $c=81.04$, $d=56.2$.

Exercise 14. Equations Involving Perimeters

1. The perimeter of a square is 96. Find a side.

2. The perimeter of a triangle is 114. The first side is 6 less than the second and 24 less than the third. Find the sides.

3. Find the dimensions of a rectangle whose perimeter is 48 if the length is 3 times the width.

4. Find the dimensions of a rectangle if its length is 4 more than the width and its perimeter is 82.

5. The length of a rectangle is 4 more than twice the width and its perimeter is $135\frac{1}{5}$. Find the length.

6. The perimeter of a rectangle is 48.648. Find the width if it is $\frac{1}{4}$ of the length.

7. The perimeter of a rectangle is 94. The width is 11.3 more than $\frac{2}{5}$ of the length. Find the length and the width.

8. The perimeter of a quadrilateral is 176. The first side is $\frac{1}{2}$ of the second, the third is 8 more than the second, and the fourth is 3 times the first. Find the sides.

Exercise 15. Area Formulas

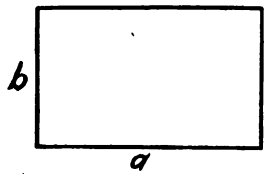


Fig. 8. Rectangle

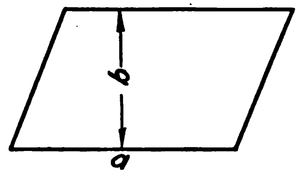


Fig. 9. Parallelogram

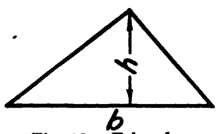


Fig. 10. Triangle

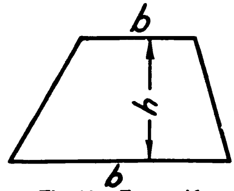


Fig. 11. Trapezoid

1. The area of a rectangle (Fig. 8) is equal to the base multiplied by the altitude. $A=a \cdot b$. Find A, if $a=11.5$, $b=18.6$.

2. Evaluate $A=a \cdot b$, if $a=2\frac{7}{8}$, $b=3\frac{3}{4}$.

3. Express the result of problem 2 in decimal form.

4. The area of a parallelogram (Fig. 9) is the base times the altitude. $A=a \cdot b$. Find A, if $a=1\frac{5}{16}$, $b=6.71$.

5. The area of a triangle (Fig. 10) is $\frac{1}{2}$ the product of the base and altitude. $A=\frac{1}{2}b \cdot h$. Find A, if $b=12.23$, $h.=6.57$.

6. Evaluate $A=\frac{1}{2}b \cdot h$, if $b=9\frac{3}{4}$, $h=4\frac{4}{5}$.

7. The area of a trapezoid (Fig. 11) is $\frac{1}{2}$ the product of the altitude and the sum of the parallel sides. $A=\frac{1}{2}h(b+b')$. Find A, if $h=10\frac{3}{4}$, $b=19\frac{2}{5}$, $b'=12\frac{1}{3}$.

8. Express the result of problem 7 in a decimal correct to .001.

Exercise 16. Circle and Circular Ring Formulas

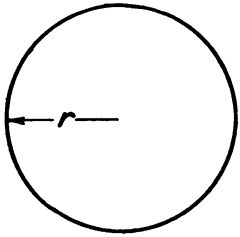


Fig. 12. Circle

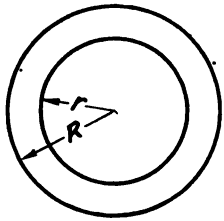


Fig. 13. Circular Ring

1. $C=2\pi r$. (Fig. 12). Find C, if $\pi=3.1416$ (See art. 17), $r=1\frac{1}{8}$.
2. $C=\pi D$. Find C, if $D=5.724$.
3. $A=\pi r^2$. Find A, if $r=1\frac{1}{8}$.
4. $A=.7854D^2$. Find A, if $D=5.724$.
5. $A=\pi(R^2-r^2)$ (Fig. 13). Find A, if $R=7\frac{1}{2}$, $r=4\frac{1}{2}$.

Exercise 17. General Formulas

Evaluate the following formulas for the values given:

1. $P=awh$, if $a=120$, $w=.32$, $h=9\frac{1}{2}$.
2. $W=\frac{l}{h}\cdot p$, if $l=25$, $h=4\frac{1}{2}$, $p=60$.
3. $F=1\frac{1}{2}d+\frac{1}{8}$, if $d=1\frac{5}{8}$.
4. $L=1\frac{3}{4}d+\frac{1}{8}$, if $d=2\frac{1}{4}$.
5. $S=\frac{1}{2}gt^2$, if $t=4$. (g is a definite number. Its value is 32.16).
6. $S=\frac{1}{2}gt^2+vt$, if $t=3$, $v=7$.
7. $D=\sqrt{a^2+b^2+c^2}$, if $a=3$, $b=4$, $c=12$.
8. $V=\frac{1}{3}h(b'+b+\sqrt{b\cdot b'})$, if $h=2\frac{3}{4}$, $b=12$, $b'=3$.
9. $F=\frac{uv}{u+v}$, if $u=11.5$, $v=6.5$.
10. $V=\frac{4}{3}\pi r^3$, if $r=2.3$.

Checking Equations

28 The solution of an equation may be tested by *evaluating* its members for the value of the unknown quantity found. If its members reduce to the same number, the value of the unknown is correct.

Example: $2x+\dfrac{2(3x-1)}{5}=3x+1.$

$$2x+\frac{6x-2}{5}=3x+1. \qquad \text{Why?}$$

$$10x+6x-2=15x+5. \qquad \text{Why?}$$

$$x=7. \qquad \text{Why?}$$

Check:

$$2\cdot 7+\frac{2(3\cdot 7-1)}{5}=3\cdot 7+1.$$

$$14+8=21+1.$$

$$22=22.$$

Exercise 18

Solve and check:

1. $6y-7=3y+20.$
2. $11=3x+9.$
3. $\dfrac{x}{3}-1=\dfrac{x}{2}-2.$
4. $2(2x+5)=13.$
5. $6(z-6)=z+8.$
6. $\dfrac{3x+11}{5}=3.$
7. $\dfrac{2(x+2)}{3}=7.$
8. $\dfrac{7(s+3)}{12}-\dfrac{s}{6}=\dfrac{s}{4}+2.$
9. $2x-1=\frac{7}{3}(5-x)-1\frac{5}{6}.$

   Note: $\frac{7}{3}(5-x)=\dfrac{7(5-x)}{3}$
10. $\dfrac{6}{5}(x+1)+\dfrac{x+3}{4}-\dfrac{1}{5}=4\frac{1}{6}.$

CHAPTER III

THE EQUATION APPLIED TO ANGLES

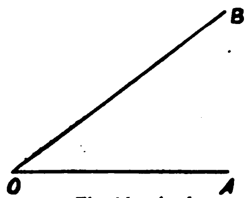


Fig. 14. Angle

29 Angle: If the line OA (Fig. 14) revolves about O as a center to the position OB, the two lines meeting at the point O form the *angle* AOB. The point O is called the *vertex* of the angle and the lines OA and OB are called the *sides* of the angle.

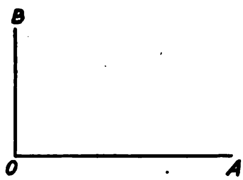


Fig. 15. Right Angle

30 Right Angle: If the line turns through one fourth of a complete revolution (Fig. 15), the angle is called a *Right Angle.*

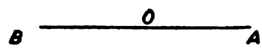


Fig. 16. Straight Angle

31 Straight Angle: If the line turns through one half of a complete revolution (Fig. 16), the angle is called a *Straight Angle.*

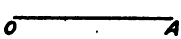


Fig. 17. Perigon

32 Perigon: If the line turns through a complete revolution (Fig. 17), returning to its original position, the angle is called a *Perigon.*

How many right angles in a straight angle?

How many right angles in a perigon?

How many straight angles in a perigon?

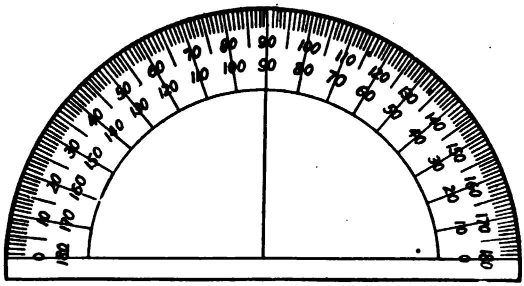


Fig. 18. Protractor

33 *A Protractor* (Fig. 18) is an instrument used for measuring and constructing angles. On it, a straight angle is divided into 180 equal parts called *degrees,* written 180°.

How many degrees in a right angle?

How many degrees in a perigon?

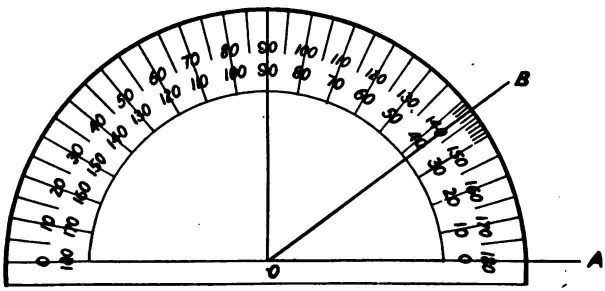


Fig. 19

Drawing Angles

34 Example: Draw an angle of 37°.

Using the straight edge of the protractor, draw a straight line OA. Place the straight edge of the protractor along the line OA, with the *center* point at O. Count 37° from the point

where the curved edge touches OA and mark the point B (Fig. 19). Again use the straight edge of the protractor to connect the points O and B.

Exercise 19

1. Draw an angle of 30°.
2. Draw an angle of 45°.
3. Draw an angle of 60°.
4. Draw an angle of 120°.
5. Draw an angle of 135°.
6. Draw an angle of 150°.
7. Draw an angle of 18°.
8. Draw an angle of 79°.
9. Draw an angle of 126°.
10. Draw an angle of 163°.

Measuring Angles

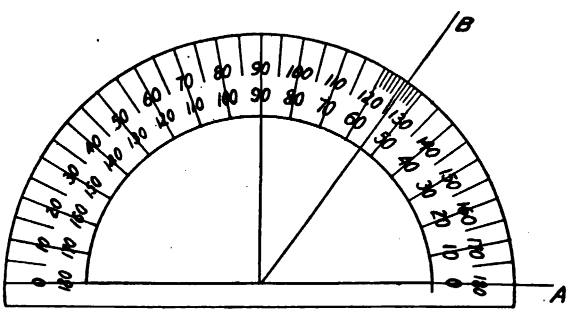
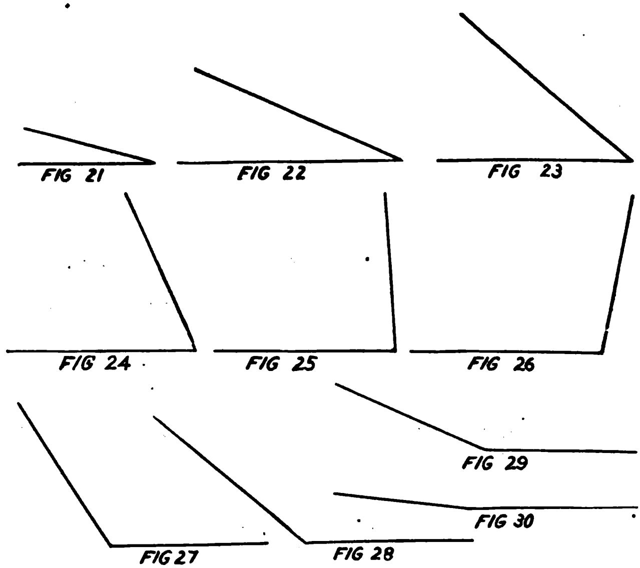


Fig. 20

35 Example: Measure the angle AOB.

Place the straight edge of the protractor along one side of the angle as OA, with its *center* at the *vertex* of the angle (Fig. 20). Count the number of degrees from the point where the curved edge of the protractor touches OA to the point where it crosses the line OB. The angle AOB contains 54°.



Exercise 20

1. Measure the angle in Fig. 21.
2. Measure the angle in Fig. 22.
3. Measure the angle in Fig. 23.
4. Measure the angle in Fig. 24.
5. Measure the angle in Fig. 25.
6. Measure the angle in Fig. 26.
7. Measure the angle in Fig. 27.
8. Measure the angle in Fig. 28.
9. Measure the angle in Fig. 29.
10. Measure the angle in Fig. 30.

Reading Angles

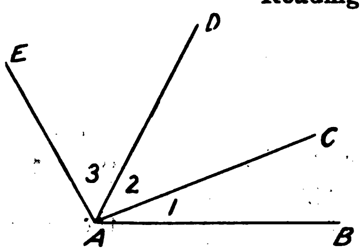


Fig. 31

36 Reading Angles: An angle is read with the letter at the vertex *between* the two letters at the ends of the sides. The angle 1 in Fig. 31 is read BAC or CAB and is written $\angle$BAC or $\angle$CAB. Read the angle $\angle 2$; $\angle 3$. (Fig. 31).

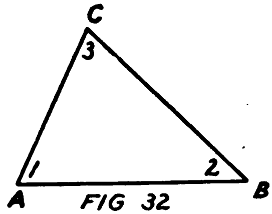


FIG 32

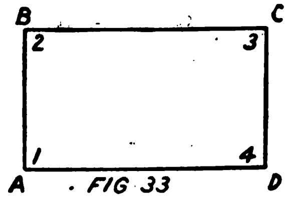


FIG 33

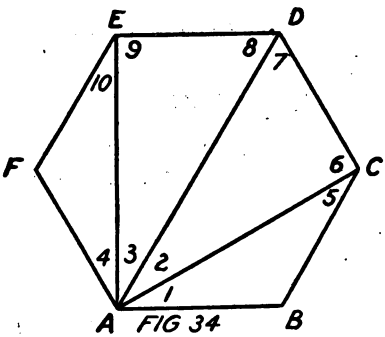


FIG 34

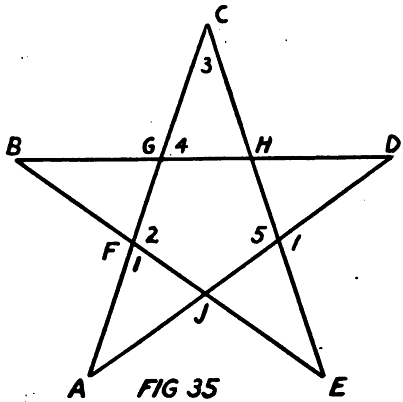


FIG 35

Exercise 21

1. Read the $\angle$s 1, 2, 3, (Fig. 32).
2. Read the $\angle$s 1, 2, 3, 4, (Fig. 33).
3. Read the $\angle$s 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, (Fig. 34).
4. Read the $\angle$s 1, 2, 3, 4, 5, (Fig. 35).

Exercise 22

1. Measure the $\angle CAD$ (Fig. 31).
2. Measure the $\angle ACB$ (Fig. 32).
3. Measure the $\angle CDA$ (Fig. 33).
4. Measure the $\angle EFA$ (Fig. 34).
5. Measure the $\angle BGF$ (Fig. 35).

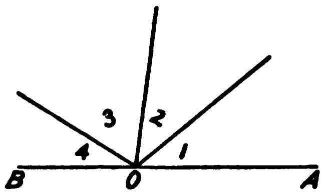


Fig. 36

37 $\angle 1 + \angle 2 + \angle 3 + \angle 4 = \angle AOB$ (Fig. 36). If AOB is a straight line, the $\angle AOB$ contains 180°. Therefore

$$\angle 1 + \angle 2 + \angle 3 + \angle 4 = 180°.$$

38 The sum of all the angles about a point on one side of a straight line is 180°.

Exercise 23

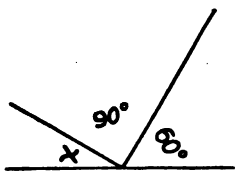


Fig. 37

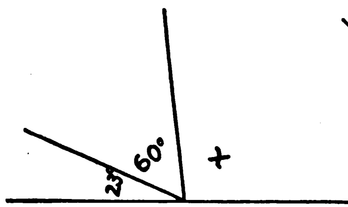


Fig. 38

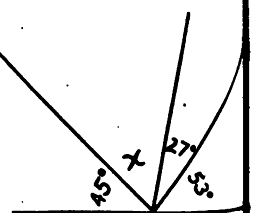


Fig. 39

1. Find x in Fig. 37. Check with a protractor.
2. Find x in Fig. 38. Check.
3. Find the unknown angle in Fig. 39. Check.
4. Three of the four angles about a point on one side of a straight line are 16°, 78°, 51°, respectively. Find the fourth angle.

5. Find the three angles about a point on one side of a straight line if the first is twice the second, and the third is three times the first.

6. Draw with a protractor the angles of problem 5 as in Figs. 37, 38, 39.

7. Find the three angles about a point on one side of a straight line if the first is twice the third, and the second is a right angle.

8. Draw the angles of problem 7.

9. Find the four angles about a point on one side of a straight line if the second is 5° less than the first, the third is 6° more than the first, and the fourth is 68°.

10. Draw the angles of problem 9.

Exercise 24

Example:

The three angles about a point on one side of a straight line are represented by $x+6°$, $\frac{4}{3}x-12°$, and $78°-\frac{x}{3}$. Find x and *the angles.*

$$x+6+\frac{4}{3}x-12+78-\frac{x}{3}=180°. \quad \text{Why?}$$

$$3x+18+4x-36+234-x=540. \quad \text{Why?}$$

$$6x+216=540. \quad \text{Why?}$$

$$6x=324. \quad \text{Why?}$$

$$x=54. \quad \text{Why?}$$

$$x+6=54+6=60° \quad \text{1st angle.}$$

$$\frac{4}{3}x-12=72-12=60° \quad \text{2nd angle.}$$

$$78-\frac{x}{3}=78-18=60° \quad \text{3rd angle.}$$

NOTE: The fact that the sum of the angles found is 180° checks the problem.

If the angles about a point on one side of a line are represented by the following, find x and the angles:

1. $\frac{5}{3}x$, $x+4$, $1\frac{1}{5}x+2$.

2. $\frac{2}{3}x-2$, $\frac{1}{11}x+7$, $3(x+7)$, $\frac{1}{3}x+19$.

3. $4(x+1)$, $7(2x-11)$, $127-6x$.

4. $3x-\frac{1}{4}$, $2x$, $2\frac{3}{4}(2x+1)$, $\frac{5}{2}(x+6)$.

5. $\frac{1}{3}x+40$, $2x-9$, $129.18-2x$.

6. Find the angles about a point on one side of a straight line if the first is 25° more than the second, and the third is three times the first.

7. Find the angles about a point on one side of a straight line if the first is 6 times the second, plus 16°, and the third is $\frac{1}{2}$ of the first, minus 4°.

8. Find the five angles about a point on one side of a straight line if the second is $\frac{1}{2}$ of the first, the third is 5° more than $\frac{2}{3}$ of the first, the fourth is 10° less than twice the first. and the fifth is $22\frac{1}{2}$°.

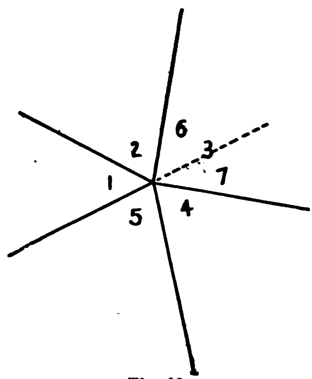


Fig. 40

39 $$\angle 1+\angle 2+\angle 6=180^\circ \text{ Why?}$$
$$\angle 7+\angle 4+\angle 5=180^\circ \text{ Why?}$$

Therefore, $\angle 1+\angle 2+\angle 3+\angle 4+\angle 5=360^\circ$.

40 *The sum of all the angles about a point is* **360°.**

Exercise 25

If all the angles about a point are represented by the following, find x and the angles:

1. $\frac{3}{5}x,\ 88-\frac{1}{3}x,\ 1\frac{1}{2}x-13,\ 4(\frac{x}{3}+1\frac{1}{2})$.
2. $23+\frac{3x}{4},\ 136-\frac{2x}{5},\ \frac{2x}{3}+93,\ \frac{x}{2}+17$.
3. $4(x-5),\ \frac{x}{2}+51\frac{1}{4},\ 3x+47\frac{1}{2}$.
4. $\frac{1}{2}(3x-36),\ \frac{1}{3}(2x+15),\ \frac{x}{6}+30,\ 82-\frac{1}{2}x,\ x+48\frac{1}{2}$.
5. $\frac{5}{3}x+3.15,\ 3(x+1.75),\ \frac{1}{2}(x+94.05)$.
6. The sum of four angles is a perigon. One is 18° more than three times the smallest, another is 59° more than the smallest, and the last is 18° less than twice the smallest. Find the four angles.

Supplementary Angles

41 *Supplementary Angles:* If the sum of two angles is a straight angle or 180°, they are called *supplementary angles. Each* is the supplement of the *other.*

Exercise 26

1. What is the supplement of 16°; 92°; 24°; $13\frac{1}{2}$°; $151\frac{3}{4}$°?
2. x is the supplement of 80°. Find x.

3. **x is the supplement of $x+32°$. Find x and its supplement.**

4. **$2x-20°$ and $7x+47°$ are supplementary angles. Find x and the angles.**

5. **One of two supplementary angles is 24° larger than the other. Find them.**

6. **The difference between two supplementary angles is 98°. Find them.**

7. **One of two supplementary angles is 4 times the other. Find the angles.**

8. **How many degrees in an angle which is the supplement of $3\frac{1}{2}$ times itself?**

9. **One of two supplementary angles is 27° less than 3 times the other. Find the angles.**

10. **One of two supplementary angles is $\frac{2}{7}$ of the sum of the other and 63°. Find the angles.**

42 **The supplement of an unknown angle may be *indicated* by $180°-x$.**

Indicate the supplement of $y°$; $d°$; $\frac{2}{3}x°$; $\frac{3}{5}y°$.

When a problem involves two supplementary angles, but is such that one is not readily expressed in terms of the other, let x equal one angle, and $180-x$ the other.

Exercise 27

1. $\frac{2}{9}$ of an angle, plus 55° is equal to $\frac{5}{9}$ of its supplement, plus 4°. Find the supplementary angles.

Let $x=$ one angle

$180-x=$ other angle

then $\frac{2}{9}x+55=\frac{5}{9}(180-x)+4.$

2. The sum of double an angle and $12\frac{1}{2}°$ is equal to $\frac{1}{2}$ the supplement of the angle. Find the supplementary angles.

3. If an angle is trebled, it is 30° more than its supplement. Find the supplementary angles.

4. If an angle is added to $\frac{1}{2}$ its supplement, the result is 128°. Find the supplementary angles.

5. If $\frac{4}{5}$ of an angle, minus 16°, is added to $\frac{2}{5}$ of its supplement, plus 72°, the result is 190°. Find the supplementary angles.

Complementary Angles

43 *Complementary Angles:* If the sum of two angles is a right angle or 90°, they are called *complementary angles.* *Each* is the complement of the *other.*

Exercise 28

1. What is the complement of 82°; 9°; 71°; $10\frac{1}{2}$°; $43\frac{7}{8}$°?

2. x is the complement of 32°. Find x.

3. x is the complement of x+76°. Find x and its complement.

4. $\frac{3}{4}x+12°$, and $\frac{2}{3}x+10°$ are complementary angles. Find x and the angles.

5. One of two complementary angles is 25° larger than the other. Find them.

6. The difference between two complementary angles is $37\frac{3}{4}$°. Find them.

7. One of two complementary angles is three times the other. Find the angles.

8. How many degrees in an angle that is the complement of $2\frac{1}{2}$ times itself?

9. One of two complementary angles is 7° more than twice the other. Find the angles.

10. One of two complementary angles is $\frac{3}{4}$ of the sum of the other and 23°. Find the angles.

44 The complement of an unknown angle may be *indicated* by $90-x$. Indicate the complement of $y°$; $m°$; $\frac{7}{8}x°$; $\frac{3}{7}y°$

• When a problem involves two complementary angles, but is such that one is not readily expressed in terms of the other, let x equal one angle, and $90-x$ the other.

Exercise 29

1. The sum of an angle and $\frac{1}{3}$ of its complement is 46°. Find the angle.

2. The complement of an angle is equal to twice the angle minus 15°. Find the angle.

3. If 20° is added to five times an angle, and 20° subtracted from $\frac{1}{5}$ of the complement, the two angles obtained, when added, will equal 114°. Find the angle.

4. $\frac{3}{7}$ of an angle is equal to $\frac{2}{3}$ of its complement, minus 14°. Find the angle.

5. $\frac{2}{5}$ of the complement of an angle, plus 15° is equal to treble the angle. Find the angle.

Exercise 30

1. The sum of $\frac{1}{4}$, $\frac{5}{6}$, and $\frac{2}{3}$ of a certain angle is 126°. Find the number of degrees in the angle.

2. The supplement of an angle is equal to four times its complement. Find the angle, its supplement and complement.

3. The sum of the supplement and complement of an angle is 98° more than twice the angle. Find the angle.

4. The complement of an angle is 20° more than $\frac{1}{3}$ of its supplement. Find the angle.

5. The sum of an angle, $\frac{1}{4}$ of the angle, its supplement, and its complement is 243°. Find the angle.

6. The complement of an angle is equal to the sum of the angle and $\frac{1}{3}$ of its supplement. Find the angle.

7. An angle increased by $\frac{1}{3}$ of its supplement is equal to twice its complement. Find the angle.

8. $\frac{4}{3}$ the supplement of an angle is equal to 3 times its complement, plus 20°. Find the angle.

9. The sum of treble an angle, $\frac{2}{3}$ of its complement, and $\frac{3}{5}$ of its supplement is equal to 62° less than a perigon. Find the angle.

10. $\frac{7}{11}$ of the complement of an angle is equal to $\frac{1}{4}$ the supplement, plus 3°. Find the angle.

11. The three angles about a point on one side of a straight line are such that the second is 89° more than $\frac{1}{6}$ of the supplement of the first, and the third is $\frac{2}{3}$ of the complement of the first. Find the three angles.

12. The sum of four angles is 223°. The second is twice the first, the third is $\frac{1}{2}$ the supplement of the second, and the fourth is the complement of the first. Find the four angles.

13. There are four angles about a point. The second is $\frac{1}{3}$ the first, the third is the supplement of the second, and the fourth is the complement of the second, plus 30°. Find the four angles.

14. There are five angles about a point on one side of a straight line. The second is $\frac{1}{5}$ of the first, the third is $\frac{1}{3}$ the supplement of the second, the fourth is $\frac{1}{3}$ the complement of the second, the fifth is 10°. Find the five angles.

15. Express by an equation that the supplement of an angle is equal to its complement, plus 90°.

Does 41° for x check the equation?

Does 25°? Does 153°? What values may x have?

CHAPTER IV

ALGEBRAIC ADDITION, SUBTRACTION, MULTIPLICATION AND DIVISION

Positive and Negative Numbers

45 **1.** The top of a mercury column of a thermometer stands at 0°. During the next hour it rises 4°, and the next 5°. What does the thermometer read at the end of the second hour?

2. The top of a mercury column stands at 0°. During the next hour it falls 4°, and the next, it falls 5°. What does it read at the end of the second hour?

3. If the mercury stands at 0°, rises 4°, and then falls 5°, what does the thermometer read?

4. If the thermometer stands at 0°, falls 4°, and then rises 5°, what does the thermometer read?

5. If the mercury stands at 0°, rises 4°, and then falls 4°, what does the thermometer read?

6. A traveler starts from a point and goes north 17 miles, and then north 15 miles. How far and in which direction is he from the starting point?

7. A traveler starts from a point and goes south 17 miles, and then south 15 miles. How far and in which direction is he from the starting point?

8. A traveler goes 17 miles south, and then 15 miles north. How far and in which direction is he from the starting point?

9. A traveler goes 17 miles north, and then 15 miles south. How far and in which direction is he from the starting point?

10. A traveler goes 17 miles south, and then 17 miles north. How far is he from the starting point?

11. An automobile travels 35 miles east, and then 40 miles east. How far and in which direction is it from the starting point?

12. An automobile travels 35 miles west and then 40 miles west. How far and in which direction is it from the starting point?

13. An automobile travels 35 miles west, and then 40 miles east. How far and in which direction is it from the starting point?

14. An automobile travels 35 miles east, and then 40 miles west. How far and in which direction is it from the starting point?

15. An automobile goes 35 miles east, and then 35 miles west. How far is it from the starting point?

16. A boy starts to work with no money. The first day he earns \$.75, and the second \$.50. How much money has he at the end of the second day?

17. A boy has to forfeit for damages \$.75 more than his wages the first day, and \$.50 more the second day. What is his financial condition at the end of the second day?

18. A boy earns \$.75 the first day, and forfeits \$.50 the second day. How much money has he?

19. A boy forfeits \$.75 the first day, and earns \$.50 the second. How much money has he?

20. A boy earns \$.75 the first day, and forfeits \$.75 the second. How much money has he?

46 Such problems as these show the necessity of making a distinction between numbers of *opposite nature.* This can be done conveniently by plus (+) and minus (−). If a number representing a certain thing is considered positive (plus), then a thing of the opposite nature *must* be negative (minus). Thus, if north 10 miles is written +10, south 10 miles *must* be written −10. If east 25 feet is written +25, west 25 feet *must* be written −25.

47 If such numbers as these are to be combined, their signs must be considered. Thus a rise of 19° in temperature followed by a rise of 9° may be expressed as follows: $(+19°) + (+9°) = +28°$. A trip 15 miles south followed by one 25 miles south may be expressed: $(-15) + (-25) = -40$. A trip 42 miles east followed by one 26 miles west is expressed: $(+42) + (-26) = +16$. A saving of $1.75 followed by an expenditure of $2.00 is expressed: $(+1.75) + (-2.00) = -.25$.

These four problems may also be written:

1.	$19+9=28$		1.	$\begin{array}{r} +19 \\ +\ 9 \\ \hline +28 \end{array}$
2.	$-15-25=-40$		2.	$\begin{array}{r} -15 \\ -25 \\ \hline -40 \end{array}$
		or		
3.	$42-26=16$		3.	$\begin{array}{r} +42 \\ -26 \\ \hline +16 \end{array}$
4.	$1.75-2.00=-.25$		4.	$\begin{array}{r} +1.75 \\ -2.00 \\ \hline -\ \ .25 \end{array}$

This combination of positive and negative numbers is called *Algebraic Addition.*

ADDITION

48 **RULE: To add two numbers with like signs, add the numbers as in arithmetic, and give to the result the common sign.**

To add two numbers with unlike signs, subtract the smaller number from the larger, and give to the result the sign of the larger.

NOTE: If no sign is expressed with a term, + is always understood. Care should be taken not to confuse this with the absence of the sign of multiplication. (See Art. 19.)

Exercise 31

Add:

1. $+19, +10$

2. $-19, -10$

3. $-19, +10$

4. $+19, -10$

5. $-10, +19$

6. $+10, -19$

7. $-75, +25$

8. $+38, +19$

9. $+11, -26$

10. $+10, -10$

11. $-40, +39$

12. $-4, +26$

13. $\frac{3}{4}, -\frac{5}{8}$

14. $\frac{3}{4}, \frac{7}{8}$

15. $\frac{11}{16}, -\frac{5}{6}$

16. $-\frac{4}{3}, \frac{3}{4}$

17. $-\frac{5}{6}, \frac{1}{4}$

18. $-3\frac{1}{2}, 2\frac{1}{3}$

19. $6\frac{2}{3}, -8\frac{5}{6}$

20. $-7\frac{4}{5}, +7\frac{3}{5}$

21. $13\frac{1}{2}, -23\frac{3}{8}$

22. $-11\frac{2}{9}, 8\frac{2}{3}$

23. $-2.32, -1.68$

24. $3.47, 5.43$

25. $8.44, -7.25$

26. $8.75, -11.25$

27. $5.732, -4.876$

28. $-18.777, -3.333$

29. $-173.29, 239.4$

30. $-208.21, 171.589$

49 1. Add -19, -10

2. Add $+11$, $+26$.

3. Add the results of problems 1 and 2.

How does the result of problem 3 compare with the result if -19, -10, $+11$, $+26$, were to be added in one problem as follows?

$$-19 - 10 + 11 + 26 = ?$$

$$-19 + 11 - 10 + 26 = ?$$

$$-19 + 26 + 11 - 10 = ?$$

$$+26 - 10 - 19 + 11 = ?$$

(See Art. 10.)

50 **RULE: To add several numbers, add all the positive numbers and all the negative numbers separately, and combine the two results.**

Exercise 32

Add:

1. $+50$, $+41$, -23, -7.
2. $+47$, -49, $+2$, -35.
3. $+3$, -40, -17, 4.
4. 82, 18, -100.
5. -79, -21, -100.
6. -119, $+1$, -21, -14, $+101$.
7. -2.36, $+4.24$, 5.73, -8.66.
8. $-3\frac{2}{5}$, $5\frac{2}{3}$, $-4\frac{7}{10}$.
9. $3\frac{1}{2}$, $7\frac{5}{6}$, $-4\frac{3}{4}$, $-5\frac{1}{3}$.
10. $23\frac{7}{8}$, $-19\frac{5}{6}$, $17\frac{2}{5}$, $-11\frac{3}{4}$, $5\frac{1}{3}$.

51 Term: A *term* is an expression whose parts are not separated by plus (+) or minus (−). $11x^2$, $-14abxy$, $+23\frac{2}{3}$ are terms.

NOTE: Such expressions as $8(x+y)$, $3(a-b)$, etc., are terms because the parts enclosed in the parenthesis are to be treated as a single quantity. (See Art. 24.)

52 Similar Terms: *Similar* or *like terms* are those which differ in their numerical coefficients only; as $2x^3yz^2$, $-\frac{7}{3}x^3yz^2$.

53 Only similar terms can be combined.

Exercise 33

Add:

1. $-16r$, $18r$, $8r$.
2. $4.2s$, $-5.7s$, $2s$.
3. $7\frac{3}{5}x$, $-4\frac{2}{5}x$, $-2\frac{1}{5}x$, x.
4. $2\frac{1}{2}ab$, $4\frac{1}{2}ab$, $-3\frac{1}{2}ab$, ab.
5. $24abc$, $-36abc$, $10abc$, $+4abc$, $-abc$.
6. $-32a^2b$, $40a^2b$, $-9a^2b$, $2a^2b$.
7. $3v^2y^3$, v^2y^3, $-9v^2y^3$, $-4v^2y^3$.
8. $-3\frac{2}{5}x^2y^3z$, $5\frac{2}{3}x^2y^3z$, $-4\frac{7}{10}x^2y^3z$.
9. $3.16xy^2z^5$, $-4.08xy^2z^5$, $6.69xy^2z^5$.
10. $8(x-y)$, $-6(x-y)$, $+4(x-y)$.
11. $-12(x+y)$, $-7(x+y)$, $-(x+y)$.
12. $-6\frac{1}{2}(c-d)$, $3\frac{2}{3}(c-d)$, $4\frac{5}{6}(c-d)$.
13. $-8(x^2+y^2)$, $24(x^2+y^2)$, $17(x^2+y^2)$, $+(x^2+y^2)$.
14. $8(x+y+z)$, $14(x+y+z)$, $-2(x+y+z)$.
15. $11(x^2+y)^4$, $-5(x^2+y)^4$, $24(x^2+y)^4$.

54 *Monomial:* An expression containing *one* term only is called a *monomial.*

55 *Polynomial:* An expression containing more than one term is called a *polynomial.* A polynomial of *two* terms is called a *binomial,* and one of *three* terms a *trinomial.*

Addition of Polynomials

56 Example: Add $2a^3-2a^2b-b^3$, $-7ab^2-11a^3$, and $b^3+7a^3+3ab^2+2a^2b$.

Since only *similar* terms can be combined, it is convenient to arrange the polynomials, one underneath the other with *similar* terms in the *same* vertical column, and add each column separately as follows:

$$\begin{array}{rrrr} 2a^3 & -2a^2b & -b^3 & \\ -11a^3 & & & -7ab^2 \\ +7a^3 & +2a^2b & +b^3 & +3ab^2 \\ \hline -2a^3 & & & -4ab^2 \end{array}$$

Exercise 34

Add:

1. $4a+3b-5c$, $-2a-m+3c$, $2m-9c+2b$, $5a+3m-4b$.
2. $pq+3qr+4rs$, $-pq+4rs-3qr$, $st-4rs$.
3. $2ax^2+3ay^2-4z^2$, $ax^2+7ay^2-4z^2$, $2z^2+ay^2-a^2x$.
4. $\frac{5}{8}a^2-\frac{1}{3}ab-\frac{1}{4}b^2$, $2b^2-a^2-\frac{2}{3}ab$, $-ab-5b^2+\frac{2}{5}a^2$.
5. $3\frac{1}{2}m-4\frac{1}{5}x+2\frac{1}{3}f$, $2\frac{1}{10}x-f+2\frac{1}{3}m$.
6. $8.75d-3.125r$, $2.873r+7.625f-10d$, $4.29f-r+1.25d$.

7. $3(x+y)-7(x-y)$, $5(x+y)+5(x-y)$, $-2(x+y)-3(x-y)$.

8. $\frac{3}{2}(a^2-b^2)-\frac{3}{4}(b^2-c^2)+\frac{4}{5}(c^2-a^2)$, $\frac{2}{3}(a^2-b^2)-\frac{5}{4}(c^2-a^2)$, $\frac{4}{3}(b^2-c^2)-2\frac{1}{6}(a^2-b^2)$.

9. $5(x+y)-7(x^2+y^2)+8(x^3+y^3)$, $-4(x^3+y^3)+5(x^2+y^2)-4(x+y)$, $2(x^2+y^2)-4(x^3+y^3)-(x+y)$.

10. $6(ab+c)+7(a-m)+a^2bc^3m$, $5a^2bc^3m-8(ab+c)-5(a-m)$, $3(ab+c)-(a-m)-4a^2bc^3m$.

SUBTRACTION

57 1. If a man is five miles north (+5) of a certain point, and another is 12 miles north (+12) of the same point, what is the difference between their positions (distance between them), and in what direction is the second from the first?

2. If the first man is 5 miles south (−5) of a point, and the second 12 miles south (−12), what is the difference between their positions, and in what direction is the second from the first?

3. The first man is at (−5), and the second is at (+12). What is the difference between their positions, and in what direction is the second from the first?

4. The first is at (+5), and the second at (−12). What is the difference between their positions, and in what direction is the second from the first?

58 To find the difference between the positions of the men in the above problem, the signs of their positions must be considered. Finding the difference between such numbers is called *Algebraic Subtraction.*

Find the difference between the positions and the direction of the second man from the first in each of the following:

Second man	+12	−12	+12	−12	+ 5	− 5	+ 5	− 5
First man	+ 5	− 5	− 5	+ 5	+12	−12	−12	+12

Add the following:

+12	−12	+12	−12	+ 5	− 5	+ 5	− 5
− 5	+ 5	+ 5	− 5	−12	+12	+12	−12

How do the results of the corresponding problems in the two groups compare?

59 **RULE: To subtract one number from another, change the sign of the subtrahend mentally and add.**

Exercise 35

Subtract:

1. $\begin{array}{r}+27\\ \underline{+12}\end{array}$	**4.** $\begin{array}{r}-32\\ \underline{+21}\end{array}$	**7.** $\begin{array}{r}+16\\ \underline{-42}\end{array}$	**10.** $\begin{array}{r}-11\frac{4}{5}\\ \underline{+15\frac{1}{5}}\end{array}$
2. $\begin{array}{r}-13\\ \underline{-\ 8}\end{array}$	**5.** $\begin{array}{r}+15\\ \underline{+82}\end{array}$	**8.** $\begin{array}{r}-\ 39\\ \underline{+100}\end{array}$	**11.** $\begin{array}{r}-\frac{3}{5}ax\\ \underline{-\frac{5}{3}ax}\end{array}$
3. $\begin{array}{r}+21\\ \underline{-\ 5}\end{array}$	**6.** $\begin{array}{r}-\ 81\\ \underline{-127}\end{array}$	**9.** $\begin{array}{r}12\frac{5}{6}\\ \underline{-4\frac{1}{3}}\end{array}$	**12.** $\begin{array}{r}7\frac{1}{5}b^2y\\ \underline{6\frac{3}{4}b^2y}\end{array}$

13. $\begin{array}{r}by^2\\ \underline{1\frac{2}{3}by^2}\end{array}$	**16.** $\begin{array}{l}3a-b+c\\ \underline{4a\quad\ \ -c}\end{array}$
14. $\begin{array}{l}-5\ (a+y)\\ \underline{-1\frac{1}{4}(a+y)}\end{array}$	**17.** $\begin{array}{l}3\frac{1}{3}x^2+5\frac{3}{4}y^3-\ z\\ \underline{2\frac{1}{4}x^2-2\frac{7}{8}y^3-2z}\end{array}$
15. $\begin{array}{l}14.92(m^2-n^2)\\ \underline{149.2\ (m^2-n^2)}\end{array}$	**18.** $\begin{array}{l}-a^3-a^2b+ab^2\\ \underline{\quad\ \ -a^2b\qquad\ \ -b^3}\end{array}$

19.
$$\begin{array}{r} .3(x+y)-4.8(x^2+y^2) \\ \underline{-5.7(x+y)+4.8(x^2+y^2)} \end{array}$$

20.
$$\begin{array}{r} -5\,(ab+c)-10.7(x+y+z)+51a^2bx^3yz+19ab^2xy^3z. \\ \underline{-3\tfrac{1}{4}(ab+c)+1.07(x+y+z)-17a^2bx^3yz+20ab^2xy^3z.} \end{array}$$

60 A problem in subtraction is often written in the form $(-19)-(+7)$. In that case it is better to *actually* change the sign of the subtrahend and then the problem is one of *addition* instead of subtraction and is written:

$$(-19)+(-7) \text{ or } -19-7=-26.$$

Exercise 36

Subtract:

1. $(-28)-(-36)$
2. $(+35)-(-21\frac{1}{2})$
3. $(-\frac{5}{6})-(+\frac{3}{7})$
4. $(+2\frac{1}{2}mx)-(+5mx)$
5. $(8.91abc)-(-3\frac{5}{8}abc)$
6. $(-2\frac{5}{16}x^2yz)-(3.1416x^2yz)$
7. $(3a+2b)-(2a+3b)$
8. $(5x^2-7y^2)-(-2x^2+y^2)$
9. $(-9\frac{2}{5}m^3n+3\frac{1}{8}mn^3)-(4\frac{5}{7}m^3n-3\frac{1}{8}mn^3)$
10. $(3\frac{1}{2}a+2b)-(3\frac{1}{2}a+7c)$
11. $(-1.7a^2b^2-2.9b^4)-(-3.3ab^3-4.16b^4)$
12. $(4x-5\frac{1}{2}y+3\frac{2}{9}z^2)-(2\frac{1}{2}x+6.25y-5\frac{1}{3}z^2)$
13. $(-11.23a^2b^2+4\frac{1}{3}b^2c^2-1\frac{7}{8}c^2a^2)-(-11.22a^2b^2+4\frac{1}{3}b^2c^2-1.875c^2a^2)$
14. $(6x^2-9mx-15m^2)-(9x^2-16m^2)$
15. $(-a^3b^2+a^2b^3+ab^4)-(a^4b-a^2b^3+b^5)$

Exercise 37. (Review)

1. From the sum of $x^2-2hx+h^2$ and $x^2-6hx+9h^2$, subtract $3x^2+2hx-4h^2$.

2. Simplify $(x^2-2xy+y^2)+(2x^2-3xy+y^2)-(3x^2-5xy+2y^2)$.

3. From $6x^3-7x-4$, subtract the sum of $9x^2-8x+x^3$ and $5-x^2+x$.

4. Simplify $(\frac{1}{2}m-\frac{1}{3}n+\frac{2}{7}p)-(\frac{3}{5}m-\frac{3}{4}n+\frac{7}{3}p)+(-\frac{7}{15}m-\frac{5}{6}n-\frac{3}{2}p)$.

5. Subtract the sum of $6-4x^3-x$ and $5x-1-2x^2$, from the sum of $2x^3+7-4x-5x^2$ and $3x^2-6x^3-2+8x$.

6. Find the difference between $(12x^4+6x^5-2)+(6x^4-8x+14-8x^3)$, and $(0)-(-10x^3+2-15x^2+11x^5-4x)$.

7. Subtract:

$$\begin{array}{r}
3x^2-5xy+2y^2-2x+7y \\
\underline{2x^2-xy+8y^2-9x-14y}
\end{array}$$

8. Simplify: $(5a^3-2a^2b+4ab^2)+(-9a^2b+7ab^2+8b^3)+(-8a^3-ab^2+2b^3)$.

9. Add:

$$\begin{array}{rrrr}
\frac{3}{2}gt^2 & +v & -\ \frac{1}{4}t & \\
gt^2 & -\frac{4}{5}v & +\ .6t & \\
-1.3gt^2 & & +\ 11t & +.02 \\
 & -10v & +1.2t & +1\frac{7}{8} \\
\hline
6gt^2 & +11.625v & & -2.25
\end{array}$$

Solve and check:

10. $12a+1-(3a-4)=2a+8+(4a+4)$.

11. $(3x-4)-6=(x-1)-(2x-3)$.

12. $-25-(-5+2p)=(13p-50)$.

13. $(3x-15)-(2x-8)=0$

14. $2k-\left(\frac{2k}{3}-\frac{7}{6}\right)=\frac{k}{2}-\left(\frac{3k}{2}-4\frac{2}{3}\right)$

15. $\frac{4}{3}x+\left(\frac{6}{5}x-\frac{2}{5}\right)-\left(\frac{5}{6}x+\frac{4}{3}\right)=1\frac{2}{3}.$

Signs of Grouping

61 *Removal of Parentheses:* By Art. 47, parentheses connected by plus signs may be used to express a problem in addition, and the parentheses can be removed without affecting the signs. For example:

$(3a-2b)+(2a-3b)=3a-2b+2a-3b=5a-5b.$

By Art. 62, two parentheses connected by a minus sign may be used to express a problem in subtraction, and the parentheses can be removed by actually changing the sign of each term enclosed in the parenthesis preceded by the minus sign (the subtrahend). For example:

$(3a-2b)-(2a+3b)=3a-2b-2a-3b=a-5b.$

62 **RULE: Parentheses preceded by minus signs may be removed if the sign of each term enclosed is changed.**

Parentheses preceded by plus signs may be removed without any change of sign.

NOTE: The sign preceding a parenthesis disappears when the parenthesis is removed.

63 Other *signs* of *grouping* often used, are the *brace* $\{\quad\}$ the *bracket* $[\quad]$ and the *vinculum* $\overline{\qquad}$. These have the same meaning as parentheses and are used to avoid confusion when several groups are needed in the same problem.

64 When several signs of grouping occur, one within the other, they are removed *one at a time*, the *innermost* one first each time.

Example: Simplify $4x-\{3x+(-2x-\overline{x-a})\}$

$$\begin{aligned}
&4x-\{3x+(-2x-\overline{x-a})\} & \\
&=4x-\{3x+(-2x-x+a)\} & \text{(removing the vinculum)}\\
&=4x-\{3x-2x-x+a\} & \text{(removing the parenthesis)}\\
&=4x-3x+2x+x-a & \text{(removing the brace)}\\
&=4x-a & \text{(combining like terms)}
\end{aligned}$$

NOTE: In the case of the vinculum, special care must be taken. $-\overline{x-a}$ is the same as $-(x-a)$. The minus sign preceding the vinculum is not the sign of x.

Exercise 38

Simplify:

1. $-6+\{5-(7+3)+12\}$
2. $10-[(7-4)-(9-7)]$
3. $4x-[2x-(x+y)+y]$
4. $-11b+[8b-(2b+b)-3b]$
5. $8kz-[7kz-\overline{3kz-5kz}]$
6. $x^2-\{3x^2-\overline{2x^2+1}\}$
7. $a^3-(-6a^2-\overline{12a+8})-(a^3+12a)$
8. $[6mn^2-(8l^3-mn^2+\overline{3n^3-mn^2})-(22mn^2-8l^3)]$
9. $3a-(5a-\{-7a+[9a-4]\})$
10. $c-[2c-(6a-b)-\{c-\overline{5a+2b}-(-5a+\overline{6a-3b})\}]$

Solve and check:

11. $4x-(5x-[3x-1])=5x-10$
12. $12x-\{8-(8x-6)-(12-3x)\}=0$
13. $-(12-20x)-\{192-(64x-36x)-12\}=96$
14. $5x-[8x-\{48-18x-(12-15x)\}]=6$
15. $12x-[-81-(-27x-\overline{4+10x})]=61-\{8x-(-20-29-4x)\}$

MULTIPLICATION

Multiplication of Monomials

Fig. 41

65 Two boys of *equal* weight are on a teeter-board at *equal* distances from the turning point (as at A and B, Fig. 41). The board balances. If one boy weighed one-half as much as the other, he would have to be twice as far from the turning point in order to balance the other. Similarly, if one weighed one-third as much as the other, he would have to be three times as far from the turning point in order to balance the other.

From these illustrations, it is readily seen that a weight of one pound, four feet from the turning point, will turn the board with *four* times as much power as a weight of one pound, *one* foot from the turning point. A weight of *three* pounds, *four* feet from the turning point, will turn the board with *three* times as much power as a weight of *one* pound, *four* feet from the turning point, and therefore with *twelve* times as much power as a weight of *one* pound, *one* foot from the turning point. The tendency of the board (*lever*) to turn under such conditions is called the *leverage*, the weights acting upon it are called *forces*, and the distance of the forces from the turning point (*fulcrum*) are called *arms*. From this explanation it is evident that:

66 The leverage caused by a force is equal to the force times the arm.

This law affords a very convenient means of working out the *law of signs* for multiplication of positive and negative numbers.

67 Let it be required to represent the product of (+5)(+4). If the result is to be thought of as a leverage, the (+5) will be the force, and the (+4) the arm. In discussing positive and negative numbers in Arts. 45, 46, and 47, measurements upward and to the right were represented by (+), and measurements downward and to the left by (−). Then the (+5) will be considered an *upward* pulling force, and the (+4) an arm measured to the *right* of the fulcrum (Fig. 42). An upward

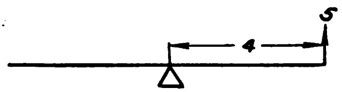


Fig. 42

force on a right arm causes the lever to turn in a counter-clockwise (opposite the hands of a clock) direction. To be consistent with arithmetic, (+5)(+4) must be (+20). Therefore, in determining the *sign* of the result of multiplication, a *counter-clockwise* motion of the lever must be *positive*, and a *clockwise* motion, *negative.*

68 Let it be required to represent the product of (−5)(−4). If the result is to be thought of as a leverage, the (−5) will be a *downward* pulling force, and the (−4), a *left* arm (Fig. 43).

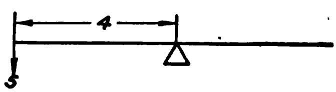


Fig. 43

It is seen that the lever turns in a *counter-clockwise* direction which is *positive.* Therefore (−5)(−4) = +20.

69 Let it be required to represent the product of $(+5)(-4)$. In this case there is an *upward*-pulling force $(+5)$, on a *left*

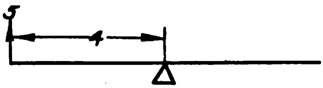


Fig. 44

arm (-4) (Fig. 44). The lever turns in a *clockwise* direction which is *negative.* Therefore $(+5)(-4)=-20$.

70 Let it be required to represent the product of $(-5)(+4)$. In this case there is a *downward*-pulling force (-5), on a *right*

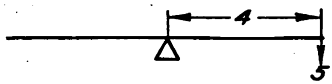


Fig. 45

arm $(+4)$ (Fig. 45). The lever turns in a *clockwise* direction which is *negative.* Therefore $(-5)(+4)=-20$.

From the four preceding articles:

1. $(+5)(+4)=+20$
2. $(-5)(-4)=+20$
3. $(+5)(-4)=-20$
4. $(-5)(+4)=-20$

From these the *law of signs for multiplication* can be derived.

71 Law of Signs for Multiplication: If two factors have like signs, their product is plus.

If two factors have unlike signs, their product is minus.

Exercise 39

Multiply:

1. $(+\frac{2}{3})(+\frac{1}{4})$
2. $(-\frac{2}{3})(-\frac{1}{4})$
3. $(+\frac{2}{3})(-\frac{1}{4})$
4. $(-\frac{2}{3})(+\frac{1}{4})$
5. $(-\frac{2}{3})(+\frac{5}{8})$
6. $(+\frac{3}{2})(-\frac{1}{2})$
7. $(-\frac{7}{8})(-\frac{3}{14})$
8. $(+3\frac{4}{5})(+1\frac{1}{5})$
9. $(-6\frac{3}{4})(-6\frac{1}{4})$
10. $(7\frac{1}{3})(-\frac{3}{11})$
11. $(-1.1)(+1.1)$
12. $(-2.03)(-4.2)$
13. $(+.3)(-.03)$
14. $(+8.75)(+3\frac{1}{5})$
15. $(-8.66)(-2\frac{1}{4})$

72 By Art. 21, x^5 means $x\cdot x\cdot x\cdot x\cdot x$ and x^3 means $x\cdot x\cdot x$.

Therefore $(x^5)(x^3)=(x\cdot x\cdot x\cdot x\cdot x)(x\cdot x\cdot x)=x^8$.

From this the *law of exponents for multiplication* can be derived.

73 Law of Exponents for Multiplication: To multiply powers of the same base, add their exponents.

NOTE: The product of powers of different bases can be indicated only. $(x^5)(y^2)=x^5y^2$.

Example: Multiply $(7a^2bx^3)$ by $(-3ab^3y^2)$

$$7a^2bx^3=7\cdot a^2\cdot b\cdot x^3$$
$$-3ab^3y^2=-3\cdot a\cdot b^3\cdot y^2$$
$$(7a^2bx^3)(-3ab^3y^2)=7\cdot a^2\cdot b\cdot x^3\cdot(-3)\cdot a\cdot b^3\cdot y^2$$

which may be arranged $7\cdot(-3)\cdot a^2\cdot a\cdot b\cdot b^3\cdot x^3\cdot y^2=-21a^3b^4x^3y^2$.

74 **RULE: To multiply monomials, multiply the numerical coefficients, and annex all the different bases, giving to each an exponent equal to the sum of the exponents of that base in the two factors.**

Exercise 40

Multiply:

1. $(3a^3)(7a^9)$
2. $(4x^4)(-6x^5)$
3. $(-5\frac{1}{2}m^{10})(-2m)$
4. $(-13a^2b)(-2ab^3)$
5. $(+5a^2bc^2)(-4ab^2c^3)$
6. $(-6a^2b)(+3b^2c)$
7. $(+2ab)(-3cd)$
8. $(+3)(-x)$
9. $(+5)(+\frac{x}{5})$
10. $(-1)(-pq)$
11. $(+2\frac{1}{2}c)(-2\frac{1}{5}d)$
12. $(-9m^2n^3)(-7m^2n^3)$
13. $(-7)(+m^2n^2)$
14. $(+7\frac{1}{3})(-gt^2)$
15. $(3xy)(xy)$
16. $(6r^2)(-\frac{1}{2}r^2)$
17. $(-x^2)(-x^3)$
18. $(-6.241)(+3.48m)$
19. $(x+y)^3 \cdot (x+y)^4$
20. $(m^2-n^2)^5 \cdot (m^2-n^2)^2$

Solve and check:

21. $(x)(3) = (-3)(-6)$
22. $(r)(-2) + (r)(+3) = (-4)(-9)$
23. $(-3)(-6) = (w)(+2) + (0)(-5)$
24. $(-2)(d) + (d)(+3) + (2)(-3) = 0$
25. $(+3)(-8) + (3k)(4) + (-2)(4k) = (0)(4)$
26. $(3s)(-6) - (-90)(-2) = (6s)(-6)$
27. $(3x)(+3) - (800)(+13) = (-5x)(19)$
28. $(5y)(-4) + (-56)(-1) = (-3y)(4)$
29. $(-9)(-2\frac{1}{9}) - (8x)(2) = (4\frac{1}{2})(6) + (-7)(4x) + (6x)(0)$
30. $(-11)(-3x) - (4\frac{1}{6})(+6) = (20\frac{1}{2})(-2) - (-3)(17x)$

75 1. What is the leverage caused by a force $+7$ on an arm -3?

2. What is the leverage caused by a force -16 on an arm $+4$?

3. What is the leverage caused by a force $+3\frac{2}{3}$ on an arm $+6$?

4. What is the leverage caused by a force -27 on an arm $-2\frac{1}{4}$?

5. What is the leverage caused by a downward force of 6, on a right arm of 3?

6. What is the leverage caused by a downward force of 12, on a left arm of 7?

7. What is the leverage caused by an upward force of 16, on a left arm of $3\frac{1}{4}$?

8. What is the leverage caused by an upward force of $3\frac{1}{3}$, on a right arm of $1\frac{1}{5}$?

76 Suppose two or more forces are acting on the lever at the same time as in Figs. 46, 47, 48.

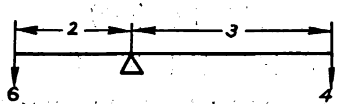


Fig. 46

What is the leverage caused by the force (-6), Fig. 46?

What is the leverage caused by the force (-4)?

In which direction will the lever turn?

This may be expressed by $(-6)(-2)+(-4)(+3)=$
$+12-12=0$

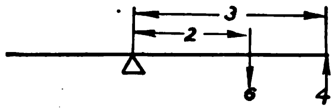


Fig. 47

What is the leverage caused by the force (-6), Fig. 47?

What is the leverage caused by the force $(+4)$?

In which direction will the lever turn?

This may be expressed by $(-6)(+2)+(+4)(+3)=-12+12=0$.

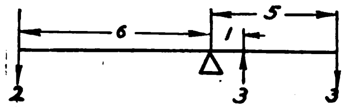


Fig. 48

What is the leverage caused by (-2), Fig. 48?

What is the leverage caused by $(+3)$?

What is the leverage caused by (-3)?

In which direction will the lever turn?

This may be expressed by $(-2)(-6)+(+3)(+1)+(-3)(+5)=12+3-15=0$.

From these illustrations the *law of leverages* may be derived.

77 *Law of Leverages: For balance, the sum of all the leverages must equal zero.*

Exercise 41

1-10. Find the unknown force or arm required for balance in the levers shown in Figs. 49, 50, 51, 52, 53, 54, 55, 56, 57, 58.

See Art. 16.

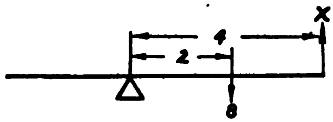


Fig. 49

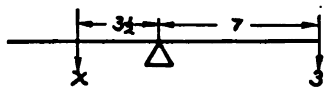


Fig. 50

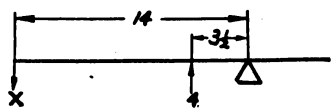


Fig. 51

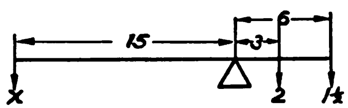


Fig. 52

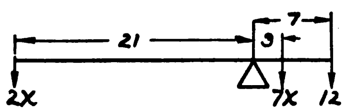


Fig. 53

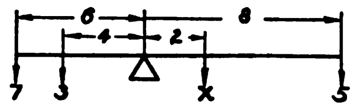


Fig. 54

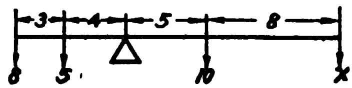


Fig. 55

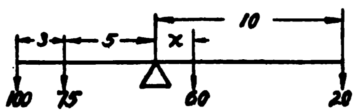


Fig. 56

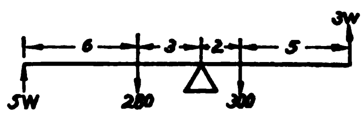


Fig. 57

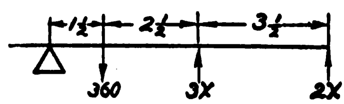


Fig. 58

11. What weight 12″ to the left of the fulcrum will balance a weight of 10 lbs., 9″ to the right of the fulcrum? (Draw a figure).

12. Two boys weighing 75 lbs. and 105 lbs. play at teeter. If the larger boy is 5′ from the fulcrum, where would the smaller boy have to sit to balance the board?

13. A crowbar is 6′ long. What weight could be raised by a man weighing 165 lbs., if the fulcrum is placed 8″ from the other end of the bar?

14. A lever 12′ long has the fulcrum at one end. How many pounds 3′ from the fulcrum can be lifted by a force of 80 lbs. at the other end?

15. A man uses an 8′ crowbar to lift a stone weighing 1600 lbs. If he thrusts the bar 1′ under the stone, with what force must he lift to raise it?

Multiplication of three or more Monomials

78 Example: Multiply $(-2a)(-3a^2)(+4a^5)(-7a^3)$

$$(-2a)(-3a^2) = +6a^3$$

$$(+6a^3)(+4a^5) = +24a^8$$

$$(+24a^8)(-7a^3) = -168a^{11}$$

$$\text{or } (-2a)(-3a^2)(+4a^5)(-7a^3) = -168a^{11}$$

Exercise 42

Multiply:

1. $(-3)(-4)(+5)$

2. $(-\frac{3}{2})(+3\frac{1}{3})(-\frac{1}{5})$

3. $(+6)(-1\frac{1}{5})(-\frac{35}{36})(-7)$

4. $(11a)(-7ab)(+4abc)(-9b^2c^2)$

5. $(a^2c^2)(-4a^2b)(-11a^3b^4)$

6. $(-4\frac{1}{2}ab)(-3\frac{2}{3}ac)(-\frac{2}{11}bc)(\frac{1}{3}abc)$
7. $(1.25m^2x)(-2.4m^3x^2y)(-4.63mxy^2)$
8. $(-3.57)(+a^7b^2)(-1\frac{3}{7}a^3b^8)$
9. $4\,(x-y)^3 \cdot (x-y)^2 \cdot \{-7\,(x-7)\}$
10. $3\,(m^2-n^3)\;\{-4(m^2-n^3)^4\} \cdot (m^2-n^3)^2$

Multiplication of Polynomials by Monomials

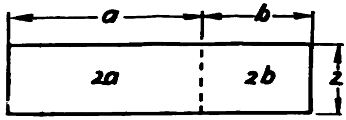


Fig. 60

79 The product of $2(a+b)$ may be represented by a rectangle (Fig. 59) having $a+b$ for one dimension and 2 for the other. The area of the entire rectangle is equal to the sum of the two rectangles, $2a$ and $2b$, or $2(a+b)=2a+2b$.

80 **RULE: To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial, and write the result as a polynomial.**

Example: Multiply $3m^2-5m+7$ by $-9m^3$.

$-9m^3(3m^2-5m+7)=-27m^5+45m^4-63m^3$.

Exercise 43

1. $a^3-7a^2b+9ab^2$ by $3a^2b^3$
2. $6x^5-5x^6-7x^4$ by $-7x^3$
3. $-3m^2-n^2+5mn$ by $4m^2n^3$
4. $a^3-2a^2x+4ax^2-8x^3$ by $-2ax$
5. $\frac{1}{4}a^2-\frac{1}{3}ab+\frac{1}{9}b^2$ by $-\frac{1}{3}b$

6. $3x^4-15x^2+24$ by $-\frac{1}{3}x^2$

7. $\frac{x}{2}+\frac{y}{3}-\frac{1}{6}$ by 12

8. $\frac{r}{10}-\frac{1}{5}+\frac{s}{6}$ by -30

9. $\frac{2m}{5}+\frac{3p}{2}-\frac{7}{15}$ by 10

Simplify:

10. $-2.4a^3z^5(2\frac{1}{3}a^2x-3\frac{1}{2}xz+.125z^3)$

11. $5(3x+2y)+4(2x-3y)$

12. $3x(4a-2y)-5x(3y-5a)$

13. $-5(3a-2)-3(a-6)+9(2a-1)$

14. $x(3x-1)-2x(7-x)-5(x^2+2x-1)$

15. $16(\frac{s}{4}+\frac{1}{8})-24(\frac{s}{3}+\frac{5}{6})$

Exercise 44

Solve and check:

1. $2(5m+1)=3(m+7)-5$
2. $3(5x+1)-4(2x+7)=3$
3. $3-(x-3)=7-2x$
4. $10(m-6)=3(m-2)-5$
5. $15y^2+11y(2-5y)+4(10y^2-9)=19$
6. $\frac{x+5}{2}-\frac{x+1}{4}=3$

SUGGESTION: $2(x+5)-(x+1)=12$ (clearing of fractions). The line of the fraction has the same meaning as a parenthesis. See Art. 62.

7. $\frac{x+5}{3}-\frac{x-10}{4}=4$

8. $\frac{1}{x-1}=4$

9. $6a-\frac{11}{a}-3(2a-1)=\frac{1}{a}$

10. $\frac{5x^2+2x+6}{5x}-x=1$

Multiplication of Polynomials by Polynomials

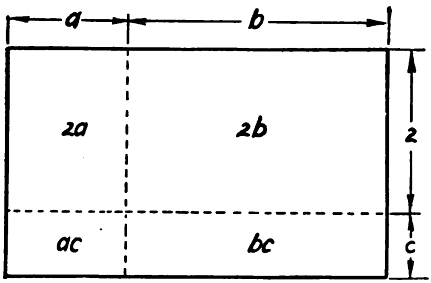


Fig. 60

81 The product of $(a+b)(c+2)$ may be represented by a rectangle whose dimensions are $a+b$ and $c+2$ (Fig. 60). The area of the entire rectangle is equal to the sum of the four rectangles, ac, bc, 2a, and 2b, or $(a+b)(c+2)=ac+bc+2a+2b$. It will be seen that the first two terms are obtained by multiplying $a+b$ by c, and the last two terms by multiplying $a+b$ by 2. It is convenient to arrange the work thus:

$$\begin{array}{r l}
a+b & \\
c+2 & \\
\hline
ac+bc & \\
 & +2a+2b \\
\hline
ac+bc & +2a+2b
\end{array}$$

82 **RULE: To multiply a polynomial by a polynomial, multiply one polynomial by each term of the other and combine like terms.**

Example: Multiply x^2-x+1 by $x-3$

$$\begin{array}{l} x^2-x\ \ +1 \\ x\ -3 \\ \hline x^3-\ x^2+\ x \\ \quad -3x^2+3x-3 \\ \hline x^3-4x^2+4x-3 \end{array}$$

Exercise 45

Multiply:

1.	$x+2$	by	$x+7$
2.	$a-3$	by	$a-5$
3.	$m+2$	by	$m-4$
4.	$3m-5n$	by	$4m+3n$
5.	$2a^2+3b^5$	by	$5a^2+4b^3$
6.	$a+1$	by	a^2-1
7.	$a-3$	by	$b+7$
8.	$2x^2-5x+7$	by	$3x-1$
9.	$4m^2-3ms-s^2$	by	m^2-3s^2
10.	$x^4-x^3+x^2-x+1$	by	$x+1$

Exercise 46

Simplify:

1. $(a-b+c)(a-b-c)$
2. $(2n^2+m^2+3mn)(2n^2-3mn+m^2)$
3. $(\frac{1}{3}x-\frac{1}{4}y)(\frac{1}{3}x+\frac{1}{4}y)$
4. $(x+3)(x-4)(x+2)$
5. $(x^2+xy+y^2)(x^2-xy+y^2)(x^2-y^2)$

6. $(2a+3b)(6a-5b)+(a-4b)(3a-b)$

7. $5(x-4)(x+1)-3(x-3)(x+2)+(x+1)(x-5)$

Solve and check:

8. $(y-5)(y+6)-(y+3)(y-4)=0$

9. $(m+3)(m+2)=(m+7)(m-5)+50$

10. $3(2x-4)(x+7)-2(3x-2)(x+5)=5-(3x-1)$

11. $\frac{4(x^2+3x+7)}{2x+7}=2x$

12. $\frac{3x^2+23}{x+3}=3x+1$

13. $\frac{(3x+2)(2x+3)}{(2x-1)(x+4)}=3$

14. $\frac{(x-2)(x-3)}{3}-\frac{(x+3)(x-4)}{4}=\frac{(x-4)(x-5)}{12}$

15. $\frac{1}{4}+\frac{x(2x+1)}{2}-\frac{(2x-3)(3x+4)}{6}=\frac{2x+17}{4}$

Exercise 47

Multiply:

1.	$2m^2-m-1$	by	$3m^2+m-2$
2.	$2p^3-3p^2q+7pq^2+4q^3$	by	$4p-3q$
3.	$a^3+b^3+ab^2+a^2b$	by	a^2b-ab^2
4.	$5-3a+7a^2$	by	$4+12a^2$
5.	$-4mn+3m^2-11n^2$	by	$2m^2-5n^2+7mn$
6.	$-5m^2+9+2m^3-4m$	by	$5m^2-1+6m$
7.	$3ax^2-4ax^3-5ax^5$	by	$1-x+2x^2$
8.	$p^3-6p^2+12p-8$	by	$p^3+6p^2+12p+8$
9.	s^3-2s^2-s-1	by	s^3+2s^2-s+1
10.	$a-1+a^3-a^2$	by	$1+a$

11. $4x^3-3x^4+2x^2-6$ by $x-x^2+1$

12. $3b^3-7b^2c+8bc^2-c^3$ by $2b^3+8b^2c-7bc^2+3c^3$

13. $a^2+b^2+c^2+ab-bc+ac$ by $a-b-c$

14. a^3-3+2a^2-a by $3-a+a^3-2a^2$

15. $\frac{1}{2}a-\frac{1}{3}b+\frac{3}{4}c-\frac{7}{8}d$ by $\frac{1}{3}a+\frac{3}{4}b-\frac{1}{2}c+\frac{7}{8}d$

16. $\frac{2}{3}a^2-\frac{3}{4}ab-\frac{4}{3}b^2$ by $1\frac{1}{2}a^2-\frac{7}{8}b^2$

17. $2\frac{2}{3}m^2n^2-4\frac{1}{2}n^3$ by $\frac{7}{8}m^2-\frac{2}{3}n$

18. $1.25a+2.375b+3.5c$ by $8a-8b+8c$

19. $.35a^2+.25ab+3.75b^2$ by $4.1a^2-.02ab-.57b^2$

20. $3.5x^2-2.1xy-1.05y^2$ by $4x-\frac{2}{7}$

Exercise 48

Simplify:

1. $(a-1)(a-2)(a-3)(a-4)$
2. $(a-b)(a^2+ab+b^2)(a^3+b^3)$
3. $(3x-4y)(2x+3y)(4x-5y)(x-7y)$
4. $(m+n)(m-n)(\frac{m^2}{2}+\frac{n^2}{3})$
5. $(x+y)(x^3+y^3)\{x^2-y(x-y)\}$
6. $(x+y+z)(x-y+z)(x+y-z)(y+z-x)$
7. $(2a+5b-c-4d)^2$
8. $(\frac{2}{3}a^3-\frac{3}{2}b^2)^3$
9. $(\frac{x}{2}-\frac{y}{3}-\frac{z}{4})^3$
10. $(x+y)(x^2-y^2)-(x-y)(x^2+y^2)$

11. $(3a-2b)(2a^2-3ab+2b^2)-3a(2a^2-3ab)$

12. $6(m-n)(m+n)-4(m^2+n^2)$

13. $12(x-y)-(x^2+x-6)(x^2+x+y)$

14. $15ab-3(2a^2+4b^2)+(3a-2b)(5a-3b)$

15. $6(a+2b-2c)^2-(2a+2b-c)^2$

16. $(x^2+1)(x-1)-(x-3)(2x-5)(x+7)-(x+2)^3$

17. $(a+b+c)^3-3(a+b+c)(a^2+b^2+c^2)$

18. $(2m^2-3mn+4n^2)^2(\frac{m^2}{2}-\frac{n^2}{3})^2$

19. $\frac{a+b+c}{2} \cdot \frac{a-b+c}{2} \cdot \frac{a+b-c}{2} \cdot \frac{b+c-a}{2}$

20. $\frac{(x-2)(2x-3)}{3} \cdot \frac{(x+2)(2x+3)}{7} \cdot \frac{(x^2+4)(4x^2+9)}{2}$

Exercise 49. (Review)

Solve and check:

1. $(-16)(-x)+(-13)(+12)+(-2)(+2x)=0$

2. $(+15)(-\frac{x}{2})+(-14)(-\frac{x}{2})+(-10\frac{5}{7})(+\frac{14}{25})=0$

3. $(-4\frac{2}{5})(5x)+(+7\frac{1}{7})(7x)+(-8\frac{4}{5})(0)+(7)(-12)=0$

4. $3x-3(\frac{1}{2}x-7)=35$

5. $(2x-1)(3x+7)-3x^2=(x-1)(3x-12)+20$

6. $\frac{3x+5}{4}=1-\frac{x-7}{6}$

7. $(\frac{1}{2}x+\frac{2}{3})(\frac{2}{3}x-\frac{1}{2})=\frac{x^2}{3}$

8. $\frac{3(3-2x)}{10}-\frac{2(x-3)}{5}+2\frac{2}{5}=\frac{4(x+4)}{5}+\frac{1}{10}$

9. $\frac{5}{3}(x+5)-\frac{7}{3}(x+7)+3\frac{5}{6}(x+1)-\frac{7}{6}(2x-5)=\frac{1}{6}(x+22)$

10. $$\frac{(x-1)(x+2)}{2}-\frac{(2x+1)(x+2)}{12}=\frac{(2x+1)(x-1)}{6}$$

11. If $\frac{4}{5}$ the supplement of an angle is subtracted from the angle, the result is 27°. Find the angle.

12. If $\frac{2}{3}$ the complement of an angle is subtracted from three times the angle, the result is 39°. Find the angle.

13. If $\frac{3}{5}$ of the supplement of an angle is decreased by $\frac{5}{8}$ of the complement, the result is 53°. Find the angle.

14. $\frac{1}{2}$ the supplement of an angle is equal to the angle diminished by $\frac{3}{4}$ of its complement. Find the angle.

15. Find three consecutive numbers such that the product of the second and third exceeds the product of the first and second by 40.

16. The difference of the squares of two consecutive numbers is 43. Find the numbers.

17. The length of a rectangle is three times its width. If its length is diminished by 6, and its width increased by 3, the area of the rectangle is unchanged. Find the dimensions.

18. Two weights, 123 and 41 respectively, are placed at the ends of a bar 24 ft. long. Where should the fulcrum be placed for balance? (Suggestion: Let $x=$ one arm, $24-x=$ the other.)

19. A man weighing 180 lbs. stands on one end of a steel rail 30 ft. long, and finds that it balances with a fulcrum placed 2 ft. from the center. What is the weight of the rail? (Suggestion: The weight of the rail may be considered a downward force at the middle point of the rail.)

20. An I-beam 32 ft. long weighing 60 lbs. per foot, is being moved by placing it upon an axle. How far from one end shall the axle be placed, if a force of $213\frac{1}{3}$ lbs. at the other end will balance it?

DIVISION

Division of Monomials

83 To divide positive and negative numbers, a *law of signs* and a *law of exponents* are necessary. These may be derived from the same laws for multiplication, from the fact that the product divided by one factor equals the other factor.

By Art. 70:

$$1.\ (+5)(+4) = +20$$
$$2.\ (-5)(-4) = +20$$
$$3.\ (+5)(-4) = -20$$
$$4.\ (-5)(+4) = -20$$

$$\text{Therefore, from 1. } \begin{cases} (+20) \div (+5) = +4 \\ (+20) \div (+4) = ? \end{cases}$$

$$\text{from 2. } \begin{cases} (+20) \div (-5) = -4 \\ (+20) \div (-4) = ? \end{cases}$$

$$\text{from 3. } \begin{cases} (-20) \div (+5) = -4 \\ (-20) \div (-4) = ? \end{cases}$$

$$\text{from 4. } \begin{cases} (-20) \div (-5) = +4 \\ (-20) \div (+4) = ? \end{cases}$$

84 Law of Signs for Division: If two numbers have like signs, their quotient is plus.

If two numbers have unlike signs, their quotient is minus.

Exercise 50

Divide:

1. $(+\frac{5}{4}) \div (+\frac{1}{5})$

2. $(-\frac{4}{5}) \div (+\frac{1}{2})$

3. $(+\frac{7}{8}) \div (-\frac{3}{4})$

4. $(-\frac{7}{8}) \div (-\frac{5}{6})$

5. $(-\frac{21}{11}) \div (+\frac{7}{33})$

6. $(+2\frac{1}{5}) \div (+3\frac{2}{3})$

7. $(-\frac{16}{21}) \div (-10)$

8. $(+7\frac{1}{5}) \div (-9)$

9. $(-42) \div (+\frac{7}{9})$

10. $(+72) \div (-4\frac{1}{2})$

11. $(-8.5)\div(-1.7)$
12. $(+3.2)\div(-.8)$
13. $(+2.65)\div(+100)$
14. $(-.008)\div(-.02)$
15. $(-15)\div(+.003)$
16. $(+3.6)\div(-2\frac{1}{4})$
17. $(-3\frac{5}{16})\div(-6.25)$
18. $(-34.56)\div(-.288)$
19. $(+26\frac{1}{4})\div(-11\frac{2}{3})$
20. $(-.0231)\div(-6\frac{3}{5})$

Exercise 51

Solve and check:

1. $3x+14-5x+15=4x+11$
2. $20x+15+32x+193-12=36x+100-32x$
3. $s(2s-3)-2s(s-7)+231=0$
4. $(x-5)(x-6)=(x-2)(x-3)$
5. $(3x-1)(4x-7)=12(x-1)^2$
6. $1\frac{2}{7}-\frac{3x}{7}-1\frac{19}{21}=\frac{4x}{3}+\frac{172}{21}$
7. $\frac{x+3}{2}-\frac{x-2}{3}=\frac{3x-5}{12}+\frac{1}{4}$
8. $\frac{2x+5}{9}-\frac{x-3}{5}-\frac{x}{3}=2x+17\frac{1}{3}$
9. $\frac{2}{3}(x+2)-\frac{8}{15}(x+5)+10=2-\frac{4}{5}(x+1)$
10. $\frac{s(s-2)}{5}-\frac{s(s-9)}{3}=\frac{-2s^2-91}{15}$

85 By Art. 72, $(x^5)(x^3)=x^8$

Therefore $(x^8)\div(x^5)=x^3$

$(x^8)\div(x^3)=?$

86 Law of Exponents for Division: To divide powers of the same base, subtract the exponent of the divisor from that of the dividend.

NOTE: The quotient of powers of different bases can be indicated only.

$$(x^5) \div (y^3) = \frac{x^5}{y^3}$$

Example: Divide $48\,a^7b^3c^2x^{10}y^4$ by $-8a^2b^2c^2x^7$

$$\frac{48a^7b^3c^2x^{10}y^4}{-8a^2b^2c^2x^7} = -6 \cdot a^5 \cdot b \cdot 1 \cdot x^3 \cdot y^4$$

$$= -6a^5bx^3y^4$$

NOTE: $\frac{c^2}{c^2} = 1$. Also $\frac{c^2}{c^2} = c^0$ by the law of exponents.

Therefore $c^0 = 1$ and may be omitted as a factor in problems like the above example.

87 **RULE: To divide a monomial by a monomial, divide the numerical coefficients, and annex all the different bases, giving to each an exponent equal to the difference of the exponents of that base in the two monomials.**

Exercise 52

Divide:

1. $-91a$ by $+13$
2. $-32x^6$ by $-8x^4$
3. $+22a^4b^3c^7$ by $-11a^2b^3c^6$
4. $+6\frac{1}{4}m^3n^3$ by $-\frac{5}{2}mn^2$
5. $-8\frac{1}{3}p^{13}q^{11}r^{21}$ by $5\frac{5}{6}q^{10}r^5$
6. $-4.24x^4y^3z^7$ by $-.4x^3y^2z^6$
7. $1.75a^7b^6x^5$ by $-.35x^4$
8. $-.85m^3n^4$ by $17n^4$
9. $-.001x^2y^3m^5$ by $-100xym$
10. $+3.1416a^2b^3m^{10}$ by $+4a^2b^3m^{10}$

Simplify:

11. $\dfrac{1155a^2x^7z^5}{-231a^2x^6z}$

12. $\dfrac{-1.732t^2u^7s^3}{+2u^6s^2}$

13. $\dfrac{3.1416x^{29}y^{14}z^{27}}{-1\frac{1}{2}x^{26}yz^{25}}$

14. $\dfrac{27m^2n^3x^7}{1.125m^2n^3x^7}$

15. $\dfrac{32.16t^2}{-.08t^2}$

16. $\dfrac{3.1416r^3}{.7854r^2}$

17. $\dfrac{+a^4b^6c^{10}}{-2a^3bc^9}$

18. $\dfrac{-a^3(x+y)^9}{+2\frac{2}{7}a(x+y)^4}$

19. $\dfrac{(a+b)^4(a+b)^4(x+y)^5}{.0625(a+b)^3(x+y)^5}$

20. $\dfrac{18.75a(m^2-n^3)^{10}}{2\frac{1}{4}(m^2-n^3)^7}$

Exercise 53

Solve for x and check:

1. $-ax=-ab$
2. $+\frac{1}{2}bx=-8b$
3. $-.3mx=2.4m$
4. $-4x=-12(a+b)$
5. $-\frac{2}{5}x=2m$
6. $3a^2b^3x=-12a^3b^3$
7. $7a-3ax=28a$
8. $4mx-7mx=12m-18m$
9. $6a^2b-7ax=-29a^2b$
10. $\dfrac{x}{3a}-\dfrac{3}{a}=\dfrac{1}{6a}$
11. $\dfrac{10x}{3m}-4m=\dfrac{5x}{6m}-\dfrac{5m}{2}$
12. $\dfrac{x+2b}{3}-\dfrac{3(x-2b)}{5}=4b$
13. $(x-5y)(x+4y)=x^2+y^2$
14. $(x+m)(x-2m)-x(x-7m)=m(3x-5m)$
15. $\dfrac{(x-3s)(x-2s)}{2}-\dfrac{x(x-5s)}{3}=\dfrac{x(x-3s)}{6}$

Division of a Polynomial by a Monomial

88 By Art. 79, $2(a+b)=2a+2b$

$$\text{Therefore, } \frac{2a.+2b}{2}=a+b$$

89 **RULE: To divide a polynomial by a monomial, divide each term of the polynomial by the monomial, and write the result as a polynomial.**

Example: Divide $21m^6-35m^4+7m^3$ by $-7m^2$

$$\frac{21m^6-35m^4+7m^3}{-7m^2}=-3m^4+5m^2-m$$

Exercise 54

Divide:

1. $\dfrac{a^2x^3c^4-ax^2c^2y^2+a^5xc^2z}{axc^2}$

2. $\dfrac{4x^3y^7z^2-12x^2y^3z^7-24x^7y^2z^3+16xyz}{-4xyz}$

3. $\dfrac{2.31m^2n^2+7.7m^3n^3-.33m^4n^4}{1.1m^2n^2}$

4. $\dfrac{-1\frac{1}{8}t^3v^2-9.81t^2v^3-.378tv^4}{-9tv^2}$

5. $\dfrac{1.125a^3x^2z^3-.375a^2x^2z^2-4.2a^2x^3z^2}{.25a^2x^2z^2}$

6. $\dfrac{-3\frac{3}{8}abcd+2\frac{1}{4}bcde+7\frac{5}{7}acde}{-1\frac{4}{5}cd}$

Solve for x and check:

7. $ax=2ab-3ac+4ae$

8. $3a^2m^3x=1.11a^3m^3-3.3a^2m^4$

9. $4m^2s^2x-3.2m^3s^2=18am^3s^2$

10. $3\frac{1}{2}xyz - 1.4y^2z = .35yz^2 - 70yz$

11. $4m^2x - 7m^3n^4 - 3m^2x + 8m^2n^5 = 5m^3n^4 - 2m^2x + 2m^2n^5$

12. $\frac{4mx}{3} - \frac{7mn+mx}{6} = -10\frac{1}{2}m^2$

13. $\frac{x}{a} - \frac{a^2b+2bx}{ab} = 3ab^2$

14. $8mn - \frac{nx}{m} + \frac{3(n^2x - m^3n^2)}{mn} = \frac{4n}{m} + m^2n$

15. $\frac{2mx+a^2m^5}{m} - \frac{5(b^3n^4+nx)}{n} = 3x - \frac{2m^2n - 3mn^2}{mn}$

Division of Polynomials by Polynomials

90 By Art. 81, $(a+b)(c+2) = ac+bc+2a+2b$

Therefore, $\frac{ac+bc+2a+2b}{a+b} = c+2$

In multiplying $a+b$ by $c+2$, the first two terms were obtained by multiplying $a+b$ by c, and the last two by multiplying $a+b$ by 2. In dividing, the c may be obtained by dividing ac by a, and the 2 may be obtained by dividing 2a by a. It is convenient to arrange the work as follows:

$$\begin{array}{r|l} & c+2 \\ \hline a+b & ac+bc+2a+2b \\ & \underline{ac+bc} \\ & \quad +2a+2b \\ & \quad \underline{+2a+2b} \end{array}$$

91 **RULE: To divide a polynomial by a polynomial, divide the first term in the dividend by the first term in the divisor to obtain the first term of the quotient. Multiply the divisor by the first term of the quotient, and subtract the result from the dividend. To obtain the other terms of the quotient, treat each remainder as a new dividend and proceed in the same way.**

Example (1): Divide $a^3-6a^2-19a+84$ by $a-7$.

$$\begin{array}{rllll}
 & a^2 & +a & -12 & \\
\cline{2-5}
a-7) & a^3 & -6a^2 & -19a & +84 \\
 & a^3 & -7a^2 & & \\
\cline{2-4}
 & & +a^2 & -19a & \\
 & & +a^2 & -7a & \\
\cline{3-4}
 & & & -12a & +84 \\
 & & & -12a & +84 \\
\cline{4-5}
\end{array}$$

Example (2): Divide $24+26x^3+120x^4-14x-111x^2$ by $-x-6+12x^2$

NOTE: Arrange the terms according to the powers of x, in both dividend and divisor.

$$\begin{array}{rlllll}
 & 10x^2 & +3x & -4 & & \\
\cline{2-6}
12x^2-x-6) & 120x^4 & +26x^3 & -111x^2 & -14x & +24 \\
 & 120x^4 & -10x^3 & -60x^2 & & \\
\cline{2-4}
 & & +36x^3 & -51x^2 & -14x & \\
 & & +36x^3 & -3x^2 & -18x & \\
\cline{3-5}
 & & & -48x^2 & +4x & +24 \\
 & & & -48x^2 & +4x & +24 \\
\cline{4-6}
\end{array}$$

Example (3): Divide $x^4+x^2y^2+y^4$ by x^2+xy+y^2.

$$\begin{array}{rlllll}
 & x^2 & -xy & +y^2 & & \\
\cline{2-6}
x^2+xy+y^2) & x^4 & & +x^2y^2 & & +y^4 \\
 & x^4 & +x^3y & +x^2y^2 & & \\
\cline{2-6}
 & & -x^3y & & & +y^4 \\
 & & -x^3y & -x^2y^2 & -xy^3 & \\
\cline{3-6}
 & & & +x^2y^2 & +xy^3 & +y^4 \\
 & & & +x^2y^2 & +xy^3 & +y^4 \\
\cline{4-6}
\end{array}$$

Exercise 55

Divide the following:

1. $x^2-7x+12$ by $x-3$
2. $a^2-2a-15$ by $a-5$
3. $a^2-3ab-28b^2$ by $a+4b$

4. $6m^4-29m^2+35$ by $2m^2-5$

5. $12x^2+31xy-15y^2$ by $x+3y$

6. $m^3+3m^2-13m-15$ by $m+1$

7. $10x^3-19x^2y+26xy^2-8y^3$ by $2x^2-3xy+4y^2$

8. $3m^4-10m^3-16m^2-10m-3$ by $3m^2+2m+1$

9. $2x^4-x^3y+4x^2y^2+xy^3+12y^4$ by $x^2-2xy+3y^2$

10. $4x^4-24x^3+51x^2-46x+15$ by $2x^2-7x+5$

11. $9x^4y-15x^3y^2+13x^2y^3-3xy^4$ by $3x^2y-xy^2$

12. $8m^7n-22m^6n^2-7m^5n^3+53m^4n^4-30m^3n^5$ by $4m^4+3m^3n-5m^2n^2$

13. $10x^3-29.3x^2y+37xy^2-20y^3$ by $2.5x^2-4.2xy+4y^2$

14. $14x^3+17x^2y+39xy^2+17y^3$ by $3.5x^2+2.5xy+8.5y^2$

15. $18x^3-53x^2+27x+14$ by $4x-8$

16. $13.12m^5n+1.36m^4n^2-7.15m^3n^3+2.35m^2n^4-.125mn^5$ by $3.2m^2n-2.4mn^2+.5n^3$

17. $a^2+ab+2ac+bc+c^2$ by $a+c$

18. $a^2bx+abcx+a^2cx+ab^2y+b^2cy+abcy$ by $ax+by$

19. $x^2+8-10x+x^3$ by $2+x^2-3x$

20. $m^4+n^4-4m^3n-4mn^3+6m^2n^2$ by m^2+n^2-2mn

21. $a^5-9a^3+7a^2-19a+10$ by a^2+3a-2

22. $16m^4-72m^2n^2+81n^4$ by $4m^2-12mn+9n^2$

23. m^3-64n^3 by $m-4n$

24. $32a^5+243b^5$ by $2a+3b$

25. $a^2+2ab+b^2-c^2$ by $a+b-c$

26. $a^2-x^2-2xy-y^2$ by $a-x-y$

27. a^4+4+3a^2 by a^2+2-a

28. $4a^2-b^2-6b-9$ by $2a+b+3$

29. $a^2-b^2+x^2-y^2+2ax+2by$ by $a+x+b-y$

30. $m^2-2mn+n^2+3m-3n+2$ by $m-n+1$

Solve and check:

31. $(a+3)x=ab+a+3b+3$

32. $(a^2-4ab+3b^2)x=a^3-8a^2b+19ab^2-12b^3$

33. $(2m-3n)x=8m^3-22m^2n+mn^2+21n^3$

34. $(a+b+2)x=a^2+2ab+b^2+4a+4b+4$

35. $(y+2)x=y^3-y^2-34y-56$

Exercise 56

1. One number is 6 more than another, and the difference of their squares is 144. Find the numbers.

2. One number is 3 less than another, and the difference of their squares is 33. Find the numbers.

3. Divide 42 into two parts such that $\frac{1}{4}$ of one is equal to $\frac{1}{3}$ of the other.

4. Divide 57 into two parts such that the sum of $\frac{1}{5}$ of the larger and $\frac{1}{4}$ of the smaller is 12.

5. The difference of two numbers is 11, and, if 18 is subtracted from $\frac{3}{4}$ of the larger, the result is $\frac{2}{7}$ of the smaller number. Find the numbers.

6. Divide 24 into two parts such that if $\frac{1}{2}$ of the smaller is subtracted from $\frac{2}{3}$ of the larger, the result is 9.

7. If the product of the first two of three consecutive numbers is subtracted from the product of the last two, the result is 18. Find the numbers.

8. If the square of the first of three consecutive numbers is subtracted from the product of the last two, the result is 41. Find the numbers.

9. I paid a certain sum of money for a lot and built a house for 3 times that amount. If the lot had cost \$240 less and the house \$280 more, the lot would have cost $\frac{1}{4}$ as much as the house. What was the cost of each?

10. A boy has $2\frac{1}{2}$ times as much money as his brother. After giving his brother \$25.00, he has only $1\frac{1}{9}$ times as much. How much had each at first?

11. The sum of $\frac{1}{2}$ a certain angle, $\frac{1}{4}$ of its complement and $\frac{1}{10}$ of its supplement is 48°. Find the angle.

12. Three times an angle, minus 4 times its complement, is equal to $\frac{2}{11}$ of its supplement + 131°. Find the angle.

13. If 3 times an angle is subtracted from $\frac{1}{2}$ its supplement, the result is $\frac{1}{11}$ of its complement.

14. A certain rectangle contains 15 sq. in. more than a square. Its length is 7 in. more and its width 3 in. less than the side of the square. Find the dimensions of the rectangle.

15. The altitude of a triangle is 4 in. more than the base, and its area exceeds one half the square of the base by 16. Find the base and altitude. (Suggestion: See Exercise 15, problem 5.)

16. A wheelbarrow is loaded with a barrel of flour weighing 196 lbs. The center of the load is 2′ from the axle of the wheel. What force at the handles, $4\frac{1}{2}'$ from the axle of the wheel, will be required to raise the load?

17. A wheelbarrow is loaded with 5 bars of pig iron weighing 77 lbs. each. How far from the axle of the wheel should the center of the load be placed, if a force of 154 lbs. 4 ft. from the axle will raise it?

18. A timber $12'' \times 18'' \times 24'$ is balanced on wheels and an axle by a force of 120 lbs. at one end. How far from the center shall the axle be placed if the timber weighs 45 lbs. per cu. ft.?

19. A lever $12'$ long weighs 24 lbs. If a weight of 30 lbs. is hung at one end and the fulcrum is placed $4'$ from this end, what force is needed at the other end for balance?

20. A piece of steel $7'$ long, weighing 15 lbs. per foot, is resting upon one end. A weight of 1400 lbs. is placed $1\frac{1}{2}'$ from that end. What force at the other end is necessary to balance the load?

CHAPTER V

RATIO, PROPORTION, AND VARIATION

Ratio

92 *Ratio:* The relation of one quantity to another of *the same kind* is called a *ratio*. *It is found by dividing the first by the second.* For example: the ratio of \$2 to \$3 is $\frac{2}{3}$, written also 2:3; the ratio of 7″ to 4″ is $\frac{7}{4}$; the ratio of 18″ to 6′ is $\frac{18}{72}=\frac{1}{4}$.

93 *Terms of Ratio:* The numerator and denominator of a ratio are respectively the first and second *terms* of a ratio. The first term of a ratio is called its *antecedent*, and the second, its *consequent*.

Exercise 57

1. Find the ratio of 85 to 51.
2. Find the ratio of 27 to 243.
3. Find the ratio of $2\frac{1}{2}$ to $3\frac{3}{4}$.
4. Find the ratio of 6.25 to 87.5.
5. Find the ratio of $\frac{3}{16}$ to .3125.
6. Find the ratio of 8″ to 6′.
7. Find the ratio of 12a to 16a.
8. Find the ratio of 5π to 3π.
9. Find the ratio of a right angle to a straight angle.
10. Find the ratio of a right angle to a perigon.
11. Find the ratio of a straight angle to a perigon.
12. Find the ratio of $\frac{2}{3}$ of a perigon to $\frac{5}{9}$ of a right angle.

13. Find the ratio of 55° to its complement.

14. Find the ratio of 55° to its supplement.

15. Find the ratio of 45° to $\frac{1}{3}$ its supplement.

16. Find the ratio of the supplement of 48° to its complement.

17. A door measures 4′ × 8′. What is the ratio of the length to the width?

18. There were 25 fair days in November, while the rest were stormy. What was the ratio of the fair to the stormy days?

19. The dimensions of two rectangles are 5″ × 8″, and 6″ × 8″. Find the ratio of their lengths, widths, perimeters, and areas.

20. The bases of two triangles are 3.9 and 2.4, and their altitudes are respectively .8 and .7. Find the ratio of their areas.

21. Find the ratio of the circumferences of two circles whose diameters are respectively $5\frac{1}{2}''$ and $2\frac{3}{4}''$. (See Exercise 16, problem 2.)

22. Find the ratio of the areas of two circles whose diameters are respectively 11″ and 13″. (See Exercise 16.)

23. Find the ratio of the two values of P in the formula $P = awh$, when $a = 120$, $w = .32$, $h = 9\frac{1}{2}$, and when $a = 48$, $w = .38$, and $h = 24$.

24. Find the ratio of the two values of F in $F = 1\frac{1}{2}d + \frac{1}{8}$, when $d = 1\frac{3}{4}$, and when $d = 2\frac{1}{8}$.

25. Find the ratio of the two values of S in $S = \frac{1}{2}gt^2$, when $t = 3\frac{1}{2}$, and when $t = 10\frac{1}{2}$. (See Exercise 17.)

94 To Express Ratios as Decimals: It is often convenient to have results in *decimal* rather than in fractional form. For example: the ratio $\frac{7}{8}$ is often written .875.

Exercise 58

Find the decimal equivalents of the following ratios:

1. $\frac{5}{8}$ 3. $\frac{9}{25}$ 5. $\frac{35}{32}$ 7. $\frac{61}{64}$ 9. $\frac{.72}{1\frac{1}{5}}$

2. $\frac{11}{16}$ 4. $\frac{19}{20}$ 6. $\frac{7\frac{1}{2}}{30}$ 8. $\frac{2\frac{3}{4}}{3\frac{2}{3}}$ 10. $\frac{3.24}{129.6}$

95 Sometimes it is sufficiently accurate to express the decimal to two places only. In this case it is necessary to determine the third place, and, if this is 5 or more, it is customary to increase the second place by 1. For example: the ratio $\frac{18}{19}=.946+$, which would be written .95 if two places only are desired.

Exercise 59

Find the decimal equivalents of the following ratios, correct to .01:

1. $\frac{9}{7}$ 2. $\frac{10}{11}$ 3. $\frac{19}{16}$ 4. $\frac{25}{7\frac{1}{8}}$ 5. $\frac{37.5}{5.15}$

Percentage is found by reducing a ratio to a decimal correct to .01, and multiplying it by 100.

For example: $\frac{.2468}{.0369}=6.688=669\%$.

6. In a class of 27 students, 22 passed an examination. Find the percentage of successful students.

7. A base ball player made 89 hits out of 321 times at bat. Find his batting average (percentage).

8. The total cost of manufacturing an article is $5.36 of which $2.79 represents labor. What per cent of the total cost is the labor?

9. If $62\frac{1}{2}$ tons of iron are obtained from 835 tons of ore, what per cent of the ore is iron?

10. In a class of students, 25 passed, 2 were conditioned, and 6 failed. Find the percentage of failures.

11. Babbitt metal is by weight 92 parts tin, 8 parts copper, and 4 parts antimony. Find the percentage of copper.

12. Potassium nitrate is composed of 39 parts of potassium, 14 parts of nitrogen, and 48 parts of oxygen. Find the percentage of potassium.

13. Potassium chloride is composed of 39 parts of potassium and 35.5 parts of chlorine. Find the percentage of chlorine.

14. Baking powder is composed of $3\frac{1}{4}$ parts of soda, $1\frac{3}{4}$ parts of cream of tartar, and 6.5 parts of starch. Find the percentage of cream of tartar.

15. If 12 quarts of water are added to 25 gallons of alcohol, what per cent of the mixture is alcohol?

16. If 5 lbs. of a substance loses 5 oz. in drying, what per cent of its original weight was water?

17. If 5 lbs. of a dried substance has lost 5 oz. in drying, what per cent of its original weight was water?

18. If a dried substance absorbs 5 oz. of water and then weighs 5 lbs., what per cent of its original weight is water?

19. The itemized cost of a house is as follows:

Masonry . .	\$ 750	Plumbing . . .	\$350
Carpenter Work	\$ 900	Furnace . . .	\$150
Lumber . .	\$1200	Painting	\$300
Plastering . .	\$ 250		

What per cent of the total cost is represented by each item?

Check by adding the per cents.

20. The population of Detroit in 1900 was 285,704, and in 1910, it was 465,776. Find the percentage of increase.

96 Specific Gravity: The *specific gravity* of a substance is the *ratio* of the *weight* of a certain volume of the substance to the *weight* of the *same* volume of water. For example: if a cubic inch of copper weighs .321 lbs., and a cubic inch of water weighs .0361 lbs., the specific gravity of copper is $\frac{.321}{.0361} = 8.88$.

Example. The dimensions of a block of cast iron are $3\frac{1}{4}'' \times 2\frac{3}{4}'' \times 1''$, and its weight is 37.5 oz. Find its specific gravity.

$$3\frac{1}{4} \times 2\frac{3}{4} \times 1 = 8.94 \text{ cu. in.} \quad \text{(the volume of the block)}$$
$$.0361 \text{ lbs.} = .5776 \text{ oz.} \quad \text{(weight of 1 cu. in. of water)}$$
$$.5776 \times 8.94 = 5.16 \quad \text{(weight of 8.94 cu. in. of water)}$$
$$\frac{37.5}{5.16} = 7.27, \quad \text{(specific gravity of iron)}$$

NOTE: Specific gravity is usually found correct to .01.

Exercise 60

1. A cubic inch of aluminum weighs .0924 lbs. Find its specific gravity.

2. A cubic inch of tungsten weighs .69 lbs. Find its specific gravity.

3. A cubic inch of cast steel weighs .282 lbs. Find its specific gravity.

4. A cubic inch of lead weighs 6.56 oz. Find its specific gravity.

5. A cubic foot of bronze weighs 550 lbs. Find its specific gravity.

6. A cubic foot of cork weighs 240 oz. Find its specific gravity.

7. A brick $2'' \times 4'' \times 8''$ weighs 4.64 lbs. Find its specific gravity.

8. A cedar block $5'' \times 3'' \times 2''$ weighs 10.5 oz. Find its specific gravity.

9. Each edge of a cubical block is $2'$. If it weighs 4450 lbs., what is its specific gravity?

10. A man weighing 185 lbs., displaces when swimming under water, 5760 cu. in. of water. Find the specific gravity of the human body.

97 Separating in a given ratio.

Example: Divide 17 into two parts which shall be in the ratio $\frac{2}{3}$.

Let $2x =$ one part.

$3x =$ other part.

NOTE $\frac{2x}{3x} = \frac{2}{3}$

Then $2x + 3x = 17$

$5x = 17$

$x = 3\frac{2}{5}$

$2x = 6\frac{4}{5}$, one part.

$3x = 10\frac{1}{5}$, other part.

Check: $6\frac{4}{5} + 10\frac{1}{5} = 17$, $\quad \frac{6\frac{4}{5}}{10\frac{1}{5}} = \frac{\frac{34}{5}}{\frac{51}{5}} = \frac{2}{3}$

Exercise 61

1. Divide 20 in the ratio $\frac{2}{3}$.

2. Divide 18 in the ratio $\frac{4}{2}$.

3. Divide 100 in the ratio $\frac{4}{1}$.

4. Divide 200 in the ratio $\frac{7}{1}$.

5. Two supplementary angles are in the ratio $\frac{4}{5}$. Find them.

6. Two complementary angles are in the ratio $\frac{7}{5}$. Find them.

7. A board 18″ long is to be divided in the ratio $\frac{7}{9}$. How far from each end is the point of division?

8. If a line 4′ 6″ long is divided in the ratio $\frac{5}{6}$, what is the length of each part?

9. Divide a legacy of $25,000 between two persons so that their shares shall be in the ratio $\frac{2}{5}$.

10. The sides of a rectangle are in the ratio $\frac{7}{3}$, and its perimeter is 100. Find the dimensions of the rectangle.

11. Bronze is composed of 11 parts tin and 39 parts copper. Find the number of pounds of tin and copper in 625 lbs. of bronze.

12. A gold medal is 18 carats fine (18 parts of pure gold in 24 parts of the whole alloy). Find the amount of pure gold in the medal if it weighs 2.7 oz.

13. Two men purchase some property together, one paying $750 and the other $450. If the property is sold for $2,000, what will be the share of each?

14. Two men agree to do a piece of work for $45. The work is completed in 10 days, but one of them was absent 2 days. How should the pay be divided?

15. How much copper would there be in 208 lbs. of Babbitt metal? (See Exercise 59, problem 11.)

16. Divide a perigon into three angles in the ratio 7 : 8 : 9.

17. Divide a line 5′ 3″ long into four parts in the ratio 5 : 6 : 7 : 3.

18. The sides of a triangle are in the ratio 5 : 8 : 9, and its perimeter is 6′ 5″. Find the sides.

19. Divide the circumference of a circle whose diameter is 16″ into three parts in the ratio 3 : 5 : 7.

20. Five angles about a point on one side of a straight line are in the ratio 1 : 2 : 3 : 4 : 5. Find them.

Proportion

98 Proportion. A *proportion* is an equation in which the two members are ratios. For example: $\frac{8}{12}=\frac{16}{24}$ is a proportion, and may be read 8 is to 12 as 16 is to 24. The first and fourth terms of a proportion are called the *extremes*, and the second and third are called the *means*. In the proportion $\frac{8}{12}=\frac{16}{24}$, 8 and 24 are the extremes, and 12 and 16, the means.

Example: Solve $\frac{5}{12}=\frac{x}{9}$

$15=4x$ (clearing of fractions.)

$x=3\frac{3}{4}$

Check: $\frac{5}{12}=\frac{3\frac{3}{4}}{9}$

$$\frac{5}{12}=\frac{\frac{15}{4}}{9}$$

$$\frac{5}{12}=\frac{5}{12}$$

Exercise 62

Solve and check:

1. $\frac{x}{25}=\frac{13}{14}$

2. $\frac{x}{7}=\frac{12}{17}$

3. $\frac{8}{x}=\frac{5}{11}$

4. $\frac{7}{9}=\frac{x}{14}$

5. $\frac{11}{12}=\frac{18}{x}$

6. $\frac{125}{x}=\frac{206}{305}$

7. $\frac{144}{195}=\frac{3x}{25}$

8. $\frac{x}{24}=\frac{3\frac{1}{2}}{4\frac{4}{5}}$

9. The ratio of x+1 to 9 is equal to the ratio of x+5 to 15. Find x.

10. The ratio of the complement of an angle to the angle is equal to the ratio $\frac{5}{7}$. Find the angle.

11. The ratio of the supplement of an angle to the angle is equal to the ratio $\frac{13}{7}$. Find the angle.

12. The ratio of an angle to 84° is equal to the ratio of its complement to 96°. Find the angle.

13. One number is 5 larger than another, and the ratio of the larger to the smaller is equal to $\frac{6}{5}$. Find the two numbers.

14. The length of a rectangle is 6 more than its width, and the ratio of the length to the width is $\frac{9}{7}$. Find the dimensions of the rectangle.

15. Two numbers are in the ratio $\frac{2}{3}$. If 2 is added to the smaller, the ratio of that number to the larger is $\frac{3}{4}$. Find the numbers. (See Example, Art. 97.)

16. If the scale of a drawing is $\frac{1}{4}''$ to 1′, how long should a line be made in the drawing to represent 32′?

17. If the scale of a drawing is $\frac{3}{4}''$ to $1'$, how long should a line be made to represent $10''$?

18. If the scale of a drawing is $1\frac{1}{2}''$ to $1'$, what line would be represented by a line $3\frac{1}{2}''$ on the drawing?

19. If a drawing is to be reduced to $\frac{5}{8}$ its size, what would be the length on the new drawing, of a dimension $3\frac{1}{2}''$ on the original drawing?

20. If a dimension line $\frac{3}{4}''$ on a drawing represents a line $4\frac{1}{2}''$ long, what is the scale of the drawing?

99 It is often necessary in shop practice to express a fraction or decimal in *halves, fourths, eighths, sixteenths,* etc. A proportion is a convenient means of changing to these denominators.

Example: How many $\frac{1}{32}$'s in $\frac{11}{15}$.

Let x = number of $\frac{1}{32}$'s in $\frac{11}{15}$.

$$\frac{x}{32}=\frac{11}{15}$$

$$15x=352$$

$$x=23\tfrac{7}{15}, \text{ approximately } 23\tfrac{1}{2}.$$

Exercise 63

1. How many $\frac{1}{8}$'s in $\frac{9}{10}$?
2. How many $\frac{1}{16}$'s in $\frac{4}{5}$?
3. How many $\frac{1}{32}$'s in .3?
4. Reduce 1.312 to eighths.
5. Reduce $1\frac{6}{35}$ to sixty-fourths.

100 Example: 1. Solve $\frac{x}{x+1}=\frac{4}{5}$

$$5x=4x+4 \left\{\text{L. C. D. is } 5\,(x+1)\right\}$$

$$x=4 \quad \text{Why?}$$

Check: $\frac{4}{4+1}=\frac{4}{5}$

$$\frac{4}{5}=\frac{4}{5}$$

Example: 2. Solve $\frac{x}{3(x-1)}=\frac{1}{6}$

$$2x=x-1 \left\{\text{L. C. D. is } 6(x-1)\right\}$$

$$x=-1 \quad \text{Why?}$$

Check: $\frac{-1}{3(-1-1)}=\frac{1}{6}$

$$\frac{-1}{-6}=\frac{1}{6}$$

$$\frac{1}{6}=\frac{1}{6}$$

Example: 3. Solve $\frac{x+1}{x+2}=\frac{x-3}{x-4}$

$$x^2-3x-4=x^2-x-6 \left\{\text{L. C. D. is } (x+2)\ (x-4)\right\}$$

$$-2x=-2 \quad \text{Why?}$$

$$x=\ \ 1 \quad \text{Why?}$$

Check: $\frac{1+1}{1+2}=\frac{1-3}{1-4}$

$$\frac{2}{3}=\frac{-2}{-3}$$

$$\frac{2}{3}=\frac{2}{3}$$

Exercise 64

Solve and check:

1. $\frac{x}{x-1}=\frac{1}{4}$

2. $\frac{x}{3(x-1)}=\frac{4}{9}$

3. $\frac{y}{5(y+2)}=\frac{1}{10}$

4. $\frac{x+5}{x-5}=\frac{1}{6}$

5. $\frac{x}{3x+1}=\frac{2}{7}$

6. $\frac{2y+3}{3y+7}=\frac{3}{4}$

7. $\frac{x}{3(5x-6)}=\frac{1}{9}$

8. $\frac{7(3x-7)}{4(x+3)}=\frac{23}{12}$

9. $\frac{2}{3x+1}=\frac{3}{5x+2}$

10. $\frac{7}{4x-3}=\frac{9}{3x+4}$

11. $\frac{x+5}{x-4}=\frac{x+25}{x-2}$

12. $\frac{5x-7}{3x-5}=\frac{10x+11}{6x+7}$

13. $\frac{2x-3}{2(x-3)}=\frac{3x}{3x-4}$

14. The ratio of an angle to its supplement is $\frac{1}{3}$. Find the angle.

15. The ratio of an angle to its complement is $\frac{2}{7}$. Find the angle.

16. The ratio of the supplement of an angle to the complement is $\frac{5}{2}$. Find the angle.

17. If an angle is increased by 3° and its complement decreased by 13°, the ratio of the two angles will then be $\frac{3}{2}$. Find the original angle.

18. The base of one rectangle is 3 less than the base of another. The altitude of the first is 3, and that of the second is 5. The ratio of the areas is $\frac{3}{7}$. Find the bases of the two rectangles.

19. The ratio of 3° to the complement of an angle is equal to the ratio of 21° to the supplement of the same angle. Find the angle.

20. Find three consecutive numbers such that the ratio of the first to the second is equal to the ratio of 5 times the third to 5 times the first plus 16.

Variation

101 Direct Proportion: If a train travels 120 miles in 3 hours, it would travel 240 miles in 6 hours. $\frac{3}{6}$ is the *ratio* of the two *times*, and $\frac{120}{240}$ is the *ratio* of the two *distances, taken in the same order.* Both ratios reduce to $\frac{1}{2}$ and therefore the problem may be expressed by the proportion, $\frac{3}{6} = \frac{120}{240}$. *An increase in time produces an increase in distance.*

If the train travels 120 miles in 3 hours, it would travel 80 miles in 2 hours because $\frac{3}{2} = \frac{120}{80}$. *A decrease in time produces a decrease in distance.*

When two quantities are so related that an increase or decrease in one produces the *same* kind of a change in the other, one is said to be *directly proportional to* the other, or to *vary directly as* the other.

Example: If a piece of steel 3 yds. long weighs 270 lbs., how much will a piece 5 yds. long weigh?

Let x = weight of the 5-yd. piece.

$\frac{3}{5} = \frac{270}{x}$ (the weight is directly proportional to the length.)

x = ?

Exercise 65

1. If 60 cu. in. of gold weighs 42 lbs., how much will 35 cu. in. weigh?

2. If the interest on a certain sum of money is \$84.20 for 5 yrs., what would be the interest for $8\frac{1}{4}$ yrs.?

3. If a section of I-beam 10 yds. long weighs 960 lbs., how long is a piece of the same material weighing 1280 lbs.?

4. An engine running at 320 revolutions per minute (R.P.M.) develops $8\frac{1}{2}$ horsepower. How many horsepower would it develop at 365 R. P. M.?

5. At 40 lbs. pressure per square inch, a given pipe discharges 180 gallons per minute. How many gallons per minute would be discharged at 55 lbs. pressure?

6. What will be the resistance of a mile of wire if the resistance of 500 yds. of the same wire is .65 ohms?

7. A steam shovel can handle 900 cu. yds. of material in 8 hrs. At the same rate how many cu. yds. can be handled in 7 hrs.?

8. A 12 pitch gear 10″ in diameter has 120 teeth. How many teeth would a 6″ gear with the same pitch have?

9. An engine running at 185 R. P. M. drives a line shaft at 210 R. P. M. At what R. P. M. should an engine run to give the line shaft a speed of 240 R. P. M.?

10. If a machine can finish 65 pieces in 75 minutes, how long will it take it to finish 104 pieces?

102 Inverse Proportion: If a train travels a given distance in 4 hrs. at the rate of 40 miles per hour, it would take 8 hrs. to travel the same distance if the rate were 20 miles per hour. $\frac{4}{8}$ is the *ratio* of the two *times*, and $\frac{40}{20}$ is the *ratio* of the two *rates, taken in the same order.* $\frac{4}{8}=\frac{1}{2}$ but $\frac{40}{20}=\frac{2}{1}$. Therefore the problem may be expressed as a proportion if one ratio is first *inverted.* $\frac{4}{8}=\frac{20}{40}$ or $\frac{8}{4}=\frac{40}{20}$. *An increase in time produces a decrease in rate.*

If a train travels a given distance in 4 hours at the rate of 40 miles per hour, it would take 2 hours to travel the same distance if the rate were 80 miles per hour, because $\frac{4}{2}=\frac{80}{40}$ or $\frac{2}{4}=\frac{40}{80}$. *A decrease in time produces an increase in rate.*

When two quantities are so related that an increase or a decrease in one produces the *opposite* kind of a change in the other, one is said to be *inversely proportional to* the other, or to *vary inversely as* the other.

Example: If 6 men can do a piece of work in 10 days, how long will it take 5 men to do it?

Let x = time it will take 5 men

$\frac{6}{5}=\frac{x}{10}$ (the number of men is inversely proportional to the number of days.)

$x=?$

Exercise 66

1. A train traveling at the rate of 50 miles per hour covers a distance in 5 hrs. How long would it take to cover the same distance if it traveled at 40 miles per hour?

2. A man walking at 4 miles per hour can travel a distance in 3 hrs. At what rate would he have to walk to cover it in 2 hrs.?

3. If 40 men can do a piece of work in 10 days, how long will it take 25 men to do it?

4. 12 men can do a piece of work in 28 days. How many men could do it in 84 days?

5. The number of posts required for a fence is 42 when they are placed 18 ft. apart. How many would be needed if they were placed 14 ft. apart?

6. One investment of \$6,000 at $3\frac{1}{2}\%$ yields the same income as another at 3%. What is the amount of the second investment?

7. A man has two investments, one of \$15,900, and the other \$21,200. The first is invested at 6%. At what rate must the other be invested to produce the same income as the first?

8. A man planned to use 36 posts spaced 9 ft. apart in building a fence. His order was 6 posts short. How far apart should he place them?

Exercise 67. (Review)

1. The circumference of a circle is directly proportional to its diameter. If the circumference of a circle whose diameter is 6″ is 18.8496″, what is the circumference of a circle whose diameter is 4″?

2. If the circumference of a circle whose diameter is 5″ is 15.708″, what is the diameter of a circle with a circumference of 28.2744″?

3. The area of a circle varies directly as the *square* of its diameter. If the area of a 2″ circle is 12.5664 sq. in., find the area of a 4″ circle. (Suggestion: $\frac{12.5664}{x} = \frac{4}{16}$)

4. The volume of a quantity of gas varies inversely as the pressure when the temperature is constant. If the volume of a gas is 600 cubic centimeters (c. c.) when the pressure is 60 grams per square centimeter, find the pressure when the volume is 150 c. c.

5. A quantity of gas measures 423 c. c. under a pressure of 815 millimeters (m. m.). What will it measure under 760 m. m.?

6. The volume of a cube varies directly as the *cube* of the edge. If the volume of an 11″ cube is 1331 cu. in., what is the volume of a 7″ cube? (See suggestion, problem 3.)

7. The volume of a sphere is directly proportional to the cube of its diameter. Find the volume of a 6″ sphere if a 10″ sphere contains 523.6 cu. in.

8. The volume of a quantity of gas varies directly as the absolute temperature when the pressure is constant. If a

quantity of gas occupies 3.25 cu. ft. when the absolute temperature is 287°, what will be its volume at 329°?

9. The velocity of a falling body varies directly as the time of falling. If the velocity acquired in 4 seconds is 128.8 ft. per sec., what would be the velocity acquired in 7 seconds?

10. The weight of a disk of copper cut from a sheet of uniform thickness varies as the square of the diameter. Find the weight of a circular piece of copper 12″ in diameter if one 7″ in diameter weighs 4.42 oz.

11. A wheel 28″ in diameter makes 42 revolutions in going a given distance. How many revolutions would a 48″ wheel make in going the same distance?

12. If 3 men can build 91 rods of fence in a certain time, how much could 7 men build in the same time?

13. If 25 men can do a piece of work in 30 days, how long would it take 27 men to do the same work?

14. If the pressure on 230 c. c. of nitrogen is changed from 760 m. m. to 665 m. m., what will be its new volume?

15. The absolute temperature of 730 c. c. of hydrogen is changed from 353° to 273°. What is its new volume?

16. If the circumference of a circle $3\frac{1}{2}''$ in diameter is 10.9956, what is the diameter of a circle whose circumference is 23.562?

17. A sum of money earns \$1750 in $3\frac{1}{2}$ yrs. How long will it take it to earn \$2750?

18. An investment of \$1125 at 5% earns the same amount as another of \$1250. What is the rate of the second investment?

19. If an investment at $2\frac{1}{4}$% produces an income of \$400, what would it produce if invested at $3\frac{3}{4}$%?

20. The diameter of a sphere which contains 47.71305 cu. in. is $4\frac{1}{2}''$. What will a sphere contain whose diameter is 3″?

CHAPTER VI

PULLEYS, GEARS, AND SPEED

103 An important problem in the running of lathes is the calculation of the speed at which the work should be turned, in order to complete the work in the shortest time possible, without injury to the work or the tools used. Similar problems arise in the use of fly-wheels, emery wheels, grindstones, etc.

104 *Rim Speed:* When the work in a lathe is turned through one complete revolution, a point upon the surface of the work travels a distance equal to the circumference of the work. In one minute, it would travel a distance equal to the circumference of the work multiplied by the number of revolutions per minute (R. P. M.).

The distance in feet traveled by a point on the circumference of a wheel in one minute is called *Rim Speed* or *Surface Speed.*

105 **RULE: To find the rim speed, multiply the circumference of the revolving object by the number of revolutions per minute (R. P. M.), and express the result in feet.**

Example 1: The diameter of a wheel is 2″. If it makes 2500 R. P. M., what is the rim speed?

$$C = \pi \cdot D = 3.1416 \cdot 2 = 6.2832''.$$

$$\frac{6.2832}{12} = .5236 \text{ (circumference in feet.)}$$

$$.5236 \times 2500 = 1309, \text{ rim speed.}$$

Example 2: The surface speed of a wheel is 3000. If the diameter is 4″, what is its R. P. M.?

$$3.1416 \cdot 4 = 12.5664''$$

$$\frac{12.5664}{12} = 1.0472 \text{ (circumference in feet.)}$$

Let x = R. P. M.

then 1.0472x = 3000

x = 2865, R. P. M.

Example 3: What is the diameter of a wheel if its R. P. M. is 2500 and its surface speed is 1500 ft. per minute?

Let x = diameter of the wheel.

Then 3.1416x = circumference of the wheel.

3.1416x · 2500 = 1500

7854x = 1500

x = .19, diameter in feet.

.19 · 12 = 2.28 diameter in inches.

Exercise 68

1. What would be the rim speed of a 12′ fly wheel running at 75 R. P. M.?

2. An emery wheel 15″ in diameter runs at 1400 R. P. M. Find the surface speed.

3. A pulley $5\frac{1}{4}$″ in diameter runs at 1250 R. P. M. What is its rim speed?

4. A 12″ circular saw runs at 2450 R. P. M. What is its cutting speed (rim speed)?

5. A 10″ emery wheel has a rim speed of 5000 ft. per minute. How many R. P. M. does it make?

6. A grindstone will stand a surface speed of 800 ft. per minute. At how many R. P. M. can it run if its diameter is 4′ 8″?

7. At how many R. P. M. should a $9\frac{1}{2}$″ shaft be turned in a lathe to give a cutting speed of 60 ft. per minute?

8. A fly wheel having a rim speed of a mile a minute runs at 120 R. P. M. What is its diameter?

9. An emery wheel runs at 950 R. P. M. If its surface speed is 5500 ft. per minute, what is its diameter?

10. A line shaft runs at 186 R. P. M. A pulley on this shaft has a rim speed of 1350 ft. per minute. What is the diameter of the pulley?

11. The splicing of a belt connecting two equal pulleys travels through the air at the rate of 2000 ft. per minute. At what speed must the pulleys run if they are 20″ in diameter?

12. A band saw runs over two pulleys each 32″ in diameter. If the band saw is 16′ long, and the speed of the wheels 600 R. P. M., what is the cutting speed of the band saw?

Pulleys

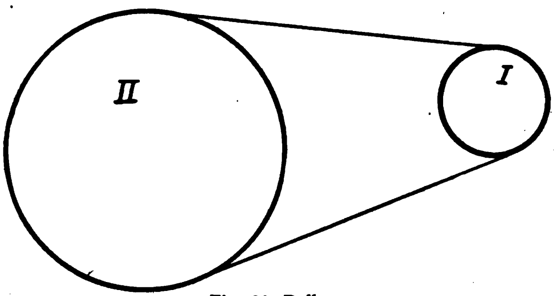


Fig. 61. Pulleys

106 When two pulleys are connected by a belt, the rim speeds of the two pulleys must be the same if there is no slipping of the belt. Suppose pulley I (Fig. 61) is 6″ in diameter and pulley II is 12″. The circumference of I is $\frac{1}{2}$ as large as the circumference of II, and therefore I will revolve twice while II revolves once. In other words, the ratio of the diameter of I to the the diameter of II is $\frac{1}{2}$, while the ratio of the R. P. M. of I to the R. P. M. of II is $\frac{2}{1}$. This may be expressed as a proportion if one ratio is inverted and therefore:

107 When two pulleys are connected by a belt, the size of the pulley varies inversely as its R. P. M.

Example: One of two pulleys connected by a belt is 12″ in diameter, and its R. P. M. is 400. What is the R. P. M. of the other pulley if it is 3″ in diameter?

Let x = R. P. M. of the second pulley.

$$\frac{12}{3}=\frac{x}{400} \quad \text{(the size varies inversely as the R. P. M.)}$$

x = 1600. R. P. M.

Exercise 69

1. A 12″ pulley running at 200 R. P. M. drives an 8″ pulley. Find the R. P. M. of the 8″ pulley.

2. A 14″ pulley drives a 26″ pulley at 175 R. P. M. What is the R. P. M. of the 14″ pulley?

3. A 30″ pulley running 240 R. P. M. is belted to a 12″ pulley. Find the R. P. M. of the 12″ pulley.

4. A pulley on a shaft running at 120 R. P. M. drives a 24″ pulley at 200 R. P. M. What is the diameter of the pulley on the shaft?

Lineshaft: The *line shaft* is the main shaft which drives the machinery of a shop by means of pulleys and belts.

Counter Shaft: A *counter shaft* is an auxiliary shaft placed between the line shaft and a machine to permit a convenient location of the machine.

5. A line shaft runs at 250 R. P. M. Determine the size of the pulley on the line shaft in order to run a 6″ pulley on a machine at 1550 R. P. M.

6. It is found necessary to run a counter shaft at 310 R. P. M. If driven by an 18″ pulley running at 175 R. P. M., what must be the diameter of the pulley on the counter shaft?

7. A counter shaft for a grinder is to be driven at 375 R. P. M. by a line shaft that runs at 210 R. P. M. If the pulley on the counter shaft is 12″ in diameter, what size pulley should be put on the line shaft?

8. A motor running at 875 R. P. M. has a $10\frac{1}{2}''$ driving pulley. If the motor drives a line shaft at 180 R. P. M., what must be the size of the line shaft pulley?

9. The diameters of two pulleys connected by a belt are in the ratio $\frac{2}{3}$. If the R. P. M. of the larger pulley is 966, what is the R. P. M. of the smaller?

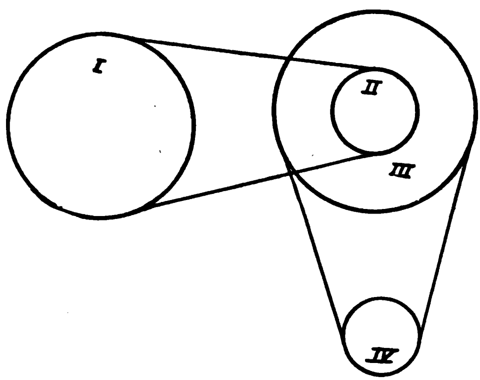


Fig. 62

10. Pulley I is belted to II, and III to IV (Fig. 62). II and III are on the same shaft. If the diameter of I is 18″ and its R. P. M. is 240, find the R. P. M. of II if its diameter is 8″. Find the R. P. M. of IV if it has a diameter of 6″, and III has one of 20″.

Step-Cone Pulleys

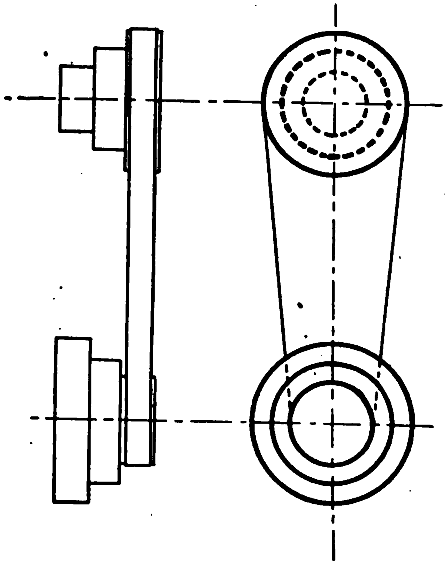


Fig. 63, Step-Cone Pulleys

108 To secure different speeds on the same machine, step-cone pulleys (Fig. 63) are used on both the driving shaft and the driven shaft. The *large* step of the driving pulley may be belted to the *small* one of the driven for *high* speed, the *medium* one to the *medium* one for *middle* speed, and the *small* one to the *large* one for *low* speed.

Example: A step-cone pulley having diameters 11″, $8\frac{1}{2}$″, and 6″, running at 120 R. P. M., drives a step-cone pulley having diameters 4″, $6\frac{1}{2}$″, and 9″. Find the three speeds.

Let x = R. P. M. at high speed.

Then $\frac{11}{4} = \frac{x}{120}$ Why?

$1320 = 4x$

$x = 330$, R. P. M. at high speed.

Let y = R. P. M. at middle speed.

$$\text{Then } \frac{8\frac{1}{2}}{6\frac{1}{2}} = \frac{y}{120} \quad \text{Why?}$$

$$1020 = 6\tfrac{1}{2}y.$$

$$y = 157-, \text{ R. P. M. at middle speed.}$$

Let z = R. P. M. at low speed.

$$\text{Then } \frac{6}{9} = \frac{z}{120} \quad \text{Why?}$$

$$720 = 9z.$$

$$z = 80 \text{ R. P. M. at low speed.}$$

Exercise 70

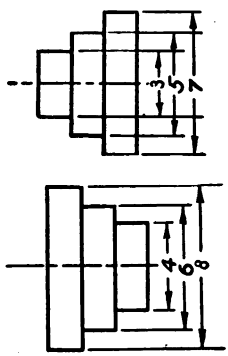


Fig 64

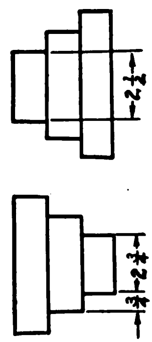


Fig. 65

1. The steps of a pair of cone pulleys are 7″, 5″, 3″, and 4″, 6″, 8″ in diameter (Fig. 64). If the lower pulley has a speed of 1050 R. P. M., find the three speeds of the upper pulley.

2. The diameters of the steps of a step-cone pulley on a machine are 10″, $8\frac{1}{2}$″ and 7″, and the corresponding counter shaft diameters are $5\frac{1}{2}$″, 7″, and $8\frac{1}{2}$″. Find the speed for each step on the machine if the counter shaft runs at 1190 R. P. M.

3. The steps of the cone pulley on a wood-turning lathe are $7\frac{1}{2}''$, $5\frac{3}{4}''$, and $4''$. The corresponding diameters of the driving pulley on the motor are $2\frac{3}{4}''$, $4\frac{1}{2}''$, and $6\frac{1}{4}''$. Find the three speeds on the lathe if the motor speed is 1165 R. P. M.

4. The smallest steps on a pair of cone pulleys are $2\frac{1}{2}''$ and $2\frac{3}{4}''$. The increase in diameter of each succeeding step is $1\frac{1}{2}''$ (Fig. 65). The first pulley has a speed of 1000 R. P. M. Find the three speeds of the second pulley.

Gears

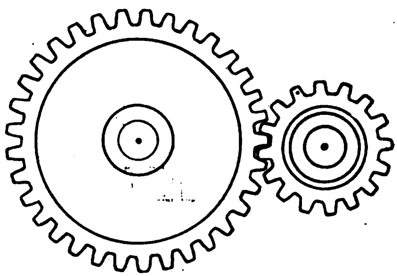


Fig. 66. Gears

109 In machines where absolute accuracy in the speed of the work is required, gears are used instead of belts to eliminate slipping. When two gears are meshed as in Fig. 66, it is evident that their rim speeds are the same. Sizes of gears are measured by the number of teeth rather than their diameters. Suppose a 48-tooth gear drives one with 24 teeth. The smaller one will revolve twice, while the larger one revolves once. The ratio of the numbers of teeth is $\frac{2}{1}$, while the ratio of the speeds is $\frac{1}{2}$. Therefore:

110 When one gear drives another, the speed is inversely proportional to the number of teeth.

Exercise 71

1. A 38-tooth gear is driving one with 72 teeth. If the first gear runs at 360 R. P. M., what is the speed of the second gear?

2. A 14-tooth gear running at 195 R. P. M. is to drive another gear at 105 R. P. M. What must be the number of teeth in the second gear?

3. Two gears are to have a speed ratio of 3 to 4. If the first gear has 36 teeth, how many will the second have?

4. The ratio of the numbers of teeth in two gears is $\frac{8}{7}$. The R. P. M. of the first is 350. What is the speed of the second?

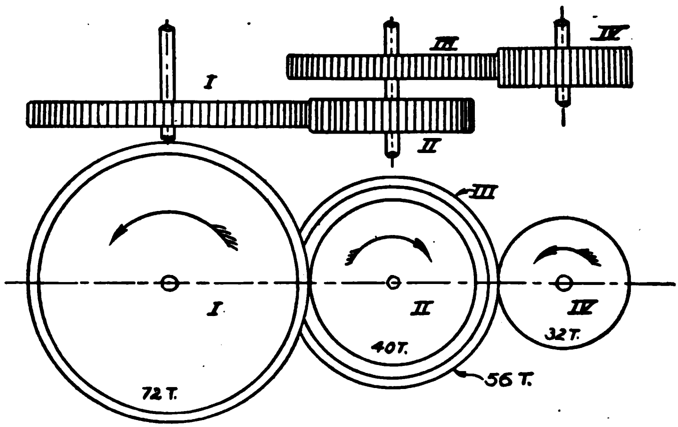


Fig. 67

5. In Fig. 67 gear I has 72 teeth, II has 40, III has 56, and IV has 32. The R. P. M. of gear I is 60. Find the R. P. M. of II. If gear III is on the same shaft as II, find the R. P. M. of IV.

Exercise 72. (Review)

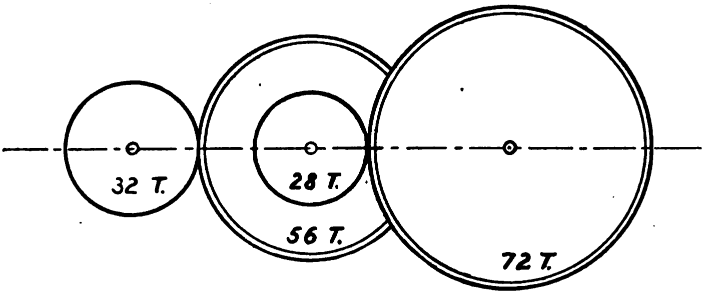


Fig. 68

1. The gear with 72 teeth has a speed of 35 R. P. M. Find the speed of the 32-tooth gear. (Fig. 68.)

2. If the 32-tooth gear (Fig. 68) is to be replaced by one which is to have a speed of 280 R. P M., what size gear must be used?

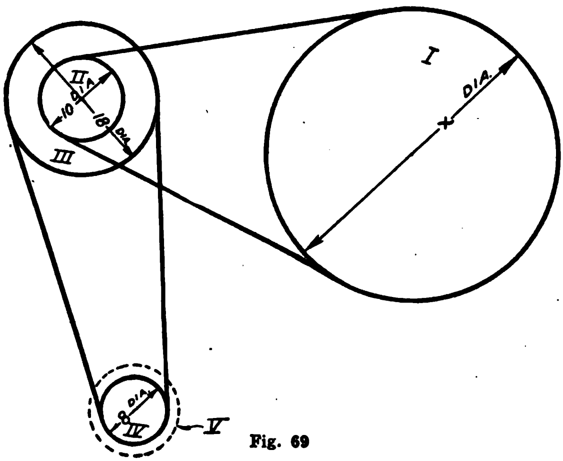


Fig. 69

3. In Fig. 69 what must be the size of the line shaft pulley (I) to run the emery wheel (V) at 1215 R. P. M., if the R. P. M. of the line shaft is 150?

4. What would be the R. P. M. of the emery wheel (V), Fig. 69, if the line shaft pulley (I) is replaced by a 48″ pulley?

5. Find the grinding speed of the emery wheel in problem 4, if its diameter is 12″.

6. A wood-turning lathe is driven by a motor running at 1200 R. P. M. The smallest step of the cone pulley on the motor shaft is 2″ in diameter, and its mate on the lathe is 7″. All increases in the diameters of succeeding steps are 2″. If the work being turned is 3″ in diameter, find the cutting speed on high speed.

7. Find the cutting speed in problem 6 on middle speed.

8. Find the cutting speed in problem 6 on low speed.

CHAPTER VII

SQUARES AND SQUARE ROOTS

111 Square of a binomial: A few kinds of multiplication problems are used so often that it is a saving of time to be able to write the result without performing the actual multiplication. One of these is the *square of a binomial.*

Find the value of the following by multiplying, and write the results as in part 1:

1. $(a+3)^2=a^2+6a+9.$	**3.** $(m+n)^2=$
2. $(b+5)^2=$	**4.** $(x+7y)^2=$

In each result, observe the following:

I. There are 3 terms in the result.

II. The first term of the result is the square of the first term of the binomial, and the third term of the result is the square of the second term of the binomial.

III. The second term of the result is 2 times the product of the two terms of the binomial.

Find the value of the following by actual multiplication and write the results as in part 1:

1. $(a-3)^2=a^2-6a+9.$	**3.** $(-m+n)^2=$
2. $(b-10)^2=$	**4.** $(-x-7y)^2=$

In each result observe that the same facts hold true as in the preceding case, *and* that the law of signs for multiplication *must* be used.

112. **RULE: To square a binomial, square the first term, take 2 times the product of the two terms, square the second term, and write the result as a trinomial.**

Example: $(2a-3bx)^2=(+2a)^2+2(+2a)(-3bx)+(-3bx)^2$
$$=(+4a^2)+(-12abx)+(+9b^2x^2)$$
$$=4a^2-12abx+9b^2x^2.$$

Exercise 73

Write the results without written multiplication:

1. $(a+1)^2$
2. $(t+u)^2$
3. $(d-4)^2$
4. $(x-y)^2$
5. $(2a+b)^2$
6. $(3x-5)^2$
7. $(a-3b)^2$
8. $(x+4y)^2$
9. $(2m+3n)^2$
10. $(5t-4u)^2$
11. $(6ab-5xy)^2$
12. $(5ab+4bx)^2$
13. $(m^2+5)^2$
14. $(x^2-8)^2$
15. $(a^2-2)^2$
16. $(a^2+b^2)^2$
17. $(2m^2-3n)^2$
18. $(4t^3-3u^2)^2$
19. $(a^4+4a)^2$
20. $(7-3m^2)^2$
21. $(m-\frac{1}{2})^2$
22. $(y+\frac{1}{3})^2$
23. $(2x-\frac{1}{2})^2$
24. $(3m+\frac{2}{3})^2$
25. $(\frac{1}{2}x+\frac{1}{3}y)^2$
26. $(\frac{3}{5}x^2-\frac{5}{3}y)^2$
27. $(1\frac{3}{4}t^2-\frac{2}{7}u^3)^2$
28. $(\frac{3}{8}f+s^5)^2$
29. $(2.3\,l-5.1m)^2$
30. $(.3125m^3n+3\frac{1}{5}mn^3)^2$
31. Square 32 mentally.

Suggestion $32^2=(30+2)^2$
$$=900+120+4=1024$$

Square the following mentally:

32. 21
33. 22
34. 29 (Suggestion $29=30-1$)
35. 34
36. 37
37. 49
38. 19
39. 35
40. 43

SQUARE ROOT

Square Root of Monomials

113 Square Root: Problems often arise in which the reverse of squaring is necessary. For example: what must be the side of a square whose area is 25 sq. in.? The side must be such that, if multiplied by itself, the result will be 25. It is evident that 5 is the side of the square since $5^2=25$.

The square root of a number is a number which if squared, will produce the given number.

Finding such a number is called *extracting square root,* and the operation is indicated by the *radical sign,* $\sqrt{}$.

$$\left.\begin{matrix}(+4)^2=16\\(-4)^2=16\end{matrix}\right\} \therefore \sqrt{16}=+4, \text{ or } -4, \text{ written } \pm 4.$$

$$\left.\begin{matrix}(-3a)^2=9a^2\\(+3a)^2=9a^2\end{matrix}\right\} \therefore \sqrt{9a^2}=\pm 3a.$$

$$(11a^2b^4)^2=121a^4b^8 \therefore \sqrt{121a^4b^8}=\pm 11a^2b^4.$$

Exercise 74

Find the square root of:

1. 81
2. 121
3. $4a^2$
4. $144m^2n^2$
5. $25x^2y^2$

Find the value of:

6. $\sqrt{49x^6y^4}$
7. $\sqrt{64a^2b^4}$
8. $\sqrt{196x^4y^6}$
9. $\sqrt{256c^{10}d^8e^4}$
10. $\sqrt{400a^2b^4c^8}$

The square root of a negative number cannot be found since, by the law of signs for multiplication, the square of either a positive or a negative number is positive.

Square Root of Trinomials

114 $\left.\begin{array}{l}(a+b)^2=a^2+2ab+b^2\\(-a-b)^2=a^2+2ab+b\end{array}\right\}\therefore\ \sqrt{a^2+2ab+b^2}=+a+b,\text{or}-a-b,$

written $\pm(a+b)$.

$\left.\begin{array}{l}(a-b)^2=a^2-2ab+b^2\\(-a+b)^2=a^2-2ab+b^2\end{array}\right\}\therefore\ \sqrt{a^2-2ab+b^2}=+a-b,\text{or}-a+b.$

written $\pm(a-b)$.

115 Trinomial Square: A trinomial in which two of the terms are squares and positive, and the other term is 2 times the product of the square roots of those terms, is called a *trinomial square,* and is the square of a binomial.

Exercise 75

Select the trinomial squares in the following:

1.	$x^2+2xy+y^2$	11.	$64a^4-176a+121$
2.	m^2-4m+4	12.	$49m^4n^2+112m^2+8n^4$
3.	m^2-4m+6	13.	$x^2+x+\frac{1}{4}$
4.	a^2-4a-4	14.	$m^2+\frac{2}{3}m+\frac{1}{9}$
5.	$x^2-6xy+9y^2$	15.	$x^2+\frac{1}{2}x+\frac{1}{8}$
6.	$4t^2+6tu+9u^2$	16.	$4a^2+ab+\frac{1}{16}b^2$
7.	$16x^2+25y^2$	17.	$9m^2-24mn+16$
8.	$169m^6-26m^3n+n^2$	18.	$x^2+\frac{3}{2}x+\frac{9}{16}$
9.	$25x^2+16y^2-40xy$	19.	$y^2+\frac{2}{5}y+\frac{4}{25}$
10.	$49^2y^2-70xyz-25z^2$	20.	$t^2-\frac{7}{8}t-\frac{49}{256}$

116 By Art. 114, $\sqrt{a^2+2ab+b^2}=\pm(a+b)$.

$\sqrt{a^2-2ab+b^2}=\pm(a-b)$.

Observe the following facts in each result:

I. The two terms of the binomial are the square roots of the two terms of the trinomial which are squares.

II. If the sign of the other term of the trinomial is plus, the terms of the binomial have like signs, and if it is minus, the terms of the binomial have unlike signs.

117 **RULE: To find the square root of a trinomial square, extract the square root of the two terms which are squares, connect them with the sign of the other term of the trinomial, and prefix the Sign $\pm$ to the binomial thus formed.**

Example: Find the square root of $25x^2+16y^2-40xy$.

$$\sqrt{25x^2+16y^2-40xy}=\pm(\sqrt{25x^2}-\sqrt{16y^2}).$$
$$=\pm(5x-4y).$$

Exercise 76

Find the square root of:

1. $9x^2-24xy+16y^2$
2. $9+6x+x^2$
3. $49m^2+14mn+n^2$
4. $t^2-10tu+25u^2$
5. $a^{16}-2a^8y^8+y^{16}$
6. $4a^6-4a^3b^2c+b^4c^2$
7. $4a^2-20ay+25y^2$
8. $9m^2+42mx+49x^2$
9. $-72xy+81x^2+16y^2$
10. $25x^6+49a^4b^2-70a^2bx^3$

Find the value of:

11. $\sqrt{30m+25+9m^2}$
12. $\sqrt{-60m^2n^2p^4+25m^4n^4+36p^8}$
13. $\sqrt{49a^4x^2+112a^3x^3+64a^2x^4}$
14. $\sqrt{x^2+x+\frac{1}{4}}$
15. $\sqrt{\frac{1}{9}a^2+25b^2-\frac{10}{3}ab}$
16. $\sqrt{\frac{4}{25}t^2+\frac{12}{5}tu+9u^2}$
17. $\sqrt{\frac{25}{36}m^4+2m^2n^2+\frac{36}{25}n^4}$
18. $\sqrt{x^4-\frac{3}{5}x^2+\frac{9}{100}}$
19. $\sqrt{\frac{4}{9}a^4b^2-\frac{4}{5}a^2bc^3+\frac{9}{25}-c^6}$
20. $\sqrt{\frac{49}{81}x^8+\frac{9}{16}y^2z^2-\frac{7}{6}x^4yz}$

Square Root of Numbers

118 By problem 31, Exercise 73.

$$32^2=(30+2)^2=900+120+4=1024.$$

$$\therefore \sqrt{1024}=\sqrt{900+120+4}=\pm(30+2)=\pm 32.$$

To extract the square root of such numbers as 1024, it is necessary to separate them into the form of a trinomial square. This can not be done by inspection. Therefore it is convenient to use the simplest form of trinomial square, $t^2+2tu+u^2$, as a formula. In that case, $t^2+2tu+u^2$ corresponds to 1024, and its square root, $t+u$, corresponds to the square root of 1024, or 32. The work may be arranged as follows:

$$\begin{array}{rr|l}
 & & t+u \\
t^2+2tu+u^2=t^2+u(2t+u)= & 1024 & \underline{30+2} \\
t^2= & 900 & \\
\hline
2t=60 & 124=u(2t+u) & \\
u=\ \ 2 & & \\
\hline
2t+u=62 & 124 & \\
\hline
\end{array}$$

$$\therefore \sqrt{1024}=\pm 32.$$

Example 1: Find the square root of 5625.

In order to find how many digits there are in the square root of a number, observe the following:

$$
\begin{aligned}
9^2 &= 81.\\
99^2 &= 9801.\\
999^2 &= 998001.
\end{aligned}
$$

The square of a number of one digit can not contain more than two digits, the square of a number of two digits can not contain more than four digits, etc. Therefore, the number of digits in the square root of a number may be determined by separating the given number into groups of two digits each, *beginning at the decimal point.*

$$
\begin{array}{rl|l}
 & & t+u \\
t^2+u(2t+u) = & 56'25 & \underline{70+5} \\
t^2 = & \underline{4900} & \\
2t = 140 & 725 = u(2t+u) & \\
u = 5 & & \\
\underline{2t+u = 145} & \underline{725} &
\end{array}
$$

$$\therefore \sqrt{5625} = \pm 75.$$

Observe that t is found by extracting the square root of the greatest square in the first group, and u is the integral number found by dividing the remainder by the number equal to $2t$.

Example 2: Find the value of $\sqrt{289}$.

$$
\begin{array}{rl|l}
 & & t+u \\
t^2+u(2t+u) = & 2'89 & \underline{10+9} \\
t^2 = & \underline{1\ 00} & \\
2t = 20 & 1\ 89 = u(2t+u) & \\
u = 9 & & \\
\underline{2t+u = 29} & \underline{2\ 61} & \\
 & ? &
\end{array}
$$

$$
\begin{array}{rl|l}
 & & \quad t+u \\
t^2+u(2t+u)=2'89 & & \underline{10+8} \\
\underline{t^2=1\,00} & & \\
2t=20 & 1\,89=u(2t+u) & \\
\underline{u=\;\;8} & & \\
\underline{2t+u=28} & \underline{2\,24} & \\
 & \quad ? &
\end{array}
$$

$$
\begin{array}{rl|l}
 & & \quad t+u \\
t^2+u(2t+u)=2'89 & & \underline{10+7} \\
\underline{t^2=1\,00} & & \\
2t=20 & 1\,89=u(2t+u) & \\
\underline{u=\;\;7} & & \\
\underline{2t+u=27} & \underline{1\,89} & \\
\end{array}
$$

$$\therefore \sqrt{289}=\pm 17.$$

Observe that in finding u, it is not always possible to take the *largest* integral number found by dividing the remainder by the number equal to $2t$.

Exercise 77

Extract the square root of:

1. 1849	**5.** 2916	**8.** 4624
2. 3136	**6.** 961	**9.** 1521
3. 576	**7.** 256	**10.** 4489
4. 5184		

Example: Find the square root of 60516.

$$
\begin{array}{rl|l}
 & & \quad t+u \\
t^2+u(2t+u)=6'05'16 & & \underline{200+40} \\
\underline{t^2=4\,00\,00} & & \\
2t=4\,00 & 2\,05\,16=u(2t+u & \\
\underline{u=\;\;40} & & \\
\underline{2t+u=4\,40} & \underline{1\,76\,00} & \\
 & \;\;29\,16 &
\end{array}
$$

The square root of 60516 will contain three digits. The first two are found in the usual way. The root is evidently 240+? and the amount that has been subtracted from 60516 (40000+17600) is 240^2. Therefore 240 may be considered a new value of t, and 2916 a new value of u(2t+u), in finding the third digit of the root. The problem then becomes:

$$
\begin{array}{rl}
 & \qquad\qquad t+u \\
t^2+u(2t+u)=6'05'16 & \quad \underline{|240+6} \\
t^2=5\ 76\ 00 & \\
\hline
2t=480 & |\,29\ 16=u(2t+u) \\
u=\ \ 6 & | \\
\hline
2t+u=486 & |\,\underline{29\ 16}
\end{array}
$$

These two operations may be combined into one problem as follows:

$$
\begin{array}{rl}
 & \qquad t+u \qquad\quad t+u \\
t^2+u(2t+u)=6'05'16 & \ \underline{|200+40}\quad \underline{240+6} \\
t^2=4\ 00\ 00 & \\
\hline
2t=400 & |\,2\ 05\ 16=u(2t+u) \\
u=\ 40 & | \\
\hline
2t+u=440 & |\,1\ 76\ 00 \\
\hline
2t=480 & |\quad 29\ 16=u(2t+u) \\
u=\ \ 6 & | \\
\hline
2t+u=486 & |\quad \underline{29\ 16}
\end{array}
$$

$$\therefore \sqrt{60516}=\pm 246.$$

Exercise 78

Find the value of:

1. $\sqrt{37636}$	**4.** $\sqrt{173889}$	**7.** $\sqrt{94249}$
2. $\sqrt{73441}$	**5.** $\sqrt{98596}$	**8.** $\sqrt{648025}$
3. $\sqrt{54756}$	**6.** $\sqrt{233289}$	**9.** $\sqrt{9778129}$
		10. $\sqrt{1022121}$

119 The operation of extracting square root may be abridged as follows:

Find the the value of:

$$\sqrt{235.6225}$$

$$
\begin{array}{r|ccccccc}
 & & & & & t & + & u \\
 & & & & t & + & u & \\
 & t & + & u & & & & \\
 & 1 & & 5. & & 3 & & 5 \\
t^2+u(2t+u)= & 2' & 3 & 5' & .6 & 2' & 2 & 5 \\
t^2= & 1 & & & & & & \\
\hline
2t=\quad 20 & 1 & 3 & 5=u(2t+u) & & & & \\
u=\quad 5 & & & & & & & \\
2t+u=\quad 25 & 1 & 2 & 5 & & & & \\
\hline
2t=\quad 300 & & 1 & 0 & 6 & 2=u(2t+u) & & \\
u=\quad 3 & & & & & & & \\
2t+u=\quad 303 & & & 9 & 0 & 9 & & \\
\hline
2t=3060 & & & 1 & 5 & 3 & 2 & 5=u(2t+u) \\
u=\quad 5 & & & & & & & \\
2t+u=3065 & & & 1 & 5 & 3 & 2 & 5 \\
\hline
\end{array}
$$

$$\therefore \sqrt{235.6225} = \pm 15.35$$

NOTE: In pointing off the given number into groups of two digits each, begin at the decimal point and proceed both right and left.

Exercise 79

Find the square root of:

1.	2323.24	**6.**	.07557001
2.	.120409	**7.**	.00003481
3.	2.6569	**8.**	1621.6729
4.	32.1489	**9.**	1040400
5.	123.4321	**10.**	1624.251204

120 If a number is not a perfect square, the operation may be continued to as many decimal places as is desired by annexing a sufficient number of ciphers.

Example: Find the value correct to .001 of:

$$\sqrt{4.32994.}$$

$$
\begin{array}{rr|llll}
 & & & & & \overline{t + u} \\
 & & & & \overline{t + u} & \\
 & & & \overline{t + u} & & \\
 & & \overline{t + u} & & & \\
 & & 2. \quad 0 & 8 & 0 & 8 = 2.081 \\
t^2+u(2t+u) = & & \overline{4. \quad 32'} & \overline{99'} & \overline{40'} & \overline{00} \\
t^2 = 4 & & & & & \\
\hline
2t = 40 & & 32 = u(2t+u) & & & \\
u = 0 & & 00 & & & \\
\hline
2t = 400 & & 32 & 99 = u(2t+u) & & \\
u = 8 & & & & & \\
2t+u = 408 & & 32 & 64 & & \\
\hline
2t = 4160 & & & 35 & 40 = u(2t+u) & \\
u = 0 & & & 00 & 00 & \\
\hline
2t = 41600 & & & 35 & 40 & 00 = u(2t+u) \\
u = 8 & & & & & \\
2t+u = 41608 & & & 33 & 28 & 64 \\
\hline
 & & & 2 & 11 & 36
\end{array}
$$

$$\therefore \sqrt{4.32994} = \pm 2.081$$

Observe that if 3 decimal places in the result are required, it is necessary to determine the digit in the 4th place, and if it is 5 or more, to add 1 to the digit in the 3rd place.

Exercise 80

Find the square root of the following correct to 4 decimal places:

1. 15
2. 38
3. 126
4. 2.5
5. 634.125

Find the value of the following correct to .0001:

6. $\sqrt{2}$
7. $\sqrt{3}$
8. $\sqrt{5}$
9. $\sqrt{.5}$
10. $\sqrt{14.4}$

$$\sqrt{36} = \sqrt{9 \cdot 4} = \sqrt{9} \cdot \sqrt{4} = 3 \cdot 2 = 6.$$

121 From this it is evident that:

The square root of a number is equal to the product of the square roots of its factors.

This law may be used to simplify the process of finding the square roots of numbers which contain one or more factors that are squares. For example:

$$\sqrt{12} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3} = \pm 3.4642.$$

Exercise 81

Given $\sqrt{2} = 1.4142$, $\sqrt{3} = 1.7321$, $\sqrt{5} = 2.2361$,

Find the value of the following correct to .001:

1. $\sqrt{8}$
2. $\sqrt{18}$
3. $\sqrt{20}$
4. $\sqrt{27}$
5. $\sqrt{32}$
6. $\sqrt{45}$
7. $\sqrt{48}$
8. $\sqrt{50}$
9. $\sqrt{72}$
10. $\sqrt{108}$

11. $\sqrt{180}$
12. $\sqrt{80}$
13. $\sqrt{125}$
14. $\sqrt{363}$
15. $\sqrt{512}$
16. $\sqrt{98}$
17. $\sqrt{147}$
18. $\sqrt{320}$
19. $\sqrt{243}$
20. $\sqrt{128}$

$(\frac{2}{3})^2=\frac{2}{3}\cdot\frac{2}{3}=\frac{4}{9}\quad\therefore\ \sqrt{\frac{4}{9}}=\pm\frac{2}{3}.$

122 The square root of a fraction is found by extracting the square root of the numerator and of the denominator.

Exercise 82

Find the square root of:

1. $\frac{16}{25}$
2. $\frac{36}{49}$
3. $\frac{1}{64}$
4. $\frac{121}{9}$
5. $\frac{169}{225}$

Find the value of the following correct to .001:

6. $\sqrt{\frac{12}{25}}$
7. $\sqrt{\frac{18}{81}}$
8. $\sqrt{\frac{5}{36}}$
9. $\sqrt{\frac{125}{144}}$
10. $\sqrt{\frac{162}{200}}$

123 In fractions where the denominator is not a perfect square the operation of finding the square root may be simplified by multiplying both numerator and denominator by a number which will make the denominator a square.

Example: $\sqrt{\frac{3}{8}}=\sqrt{\frac{6}{16}}=\frac{\sqrt{6}}{\sqrt{16}}=\frac{\pm 2.4495}{4}=\pm.6124$

Exercise 83

Find the value of the following correct to .001:

1. $\sqrt{\frac{1}{2}}$	6. $\sqrt{\frac{5}{6}}$
2. $\sqrt{\frac{1}{3}}$	7. $\sqrt{\frac{3}{5}}$
3. $\sqrt{\frac{1}{5}}$	8. $\sqrt{\frac{3}{4}}$
4. $\sqrt{\frac{1}{6}}$	9. $\sqrt{\frac{7}{8}}$
5. $\sqrt{\frac{2}{3}}$	10. $\sqrt{\frac{24}{5}}$

Quadratic Equations

124 Quadratic Equation: A *quadratic equation* is one which contains the *square of the unknown quantity* as the highest power of the unknown.

Example: $$\frac{x}{2}-\frac{13}{3x}=\frac{3x}{2}-\frac{40}{3x}$$

$$3x^2-26=9x^2-80 \quad \text{Why?}$$

$$54=6x^2 \quad \text{Why?}$$

$$x^2=9 \quad \text{Why?}$$

$$x=\pm 3 \text{ (extracting the square root of both members)}$$

Observe that:

I. A quadratic equation of the form $x^2=9$ may be transformed into one containing the first power of the unknown by extracting the square root of both members.

II. In extracting the square root of both members of the equation $x^2=9$, the full result would be $\pm x=\pm 3$, which is a condensed form of:

1. $+x=+3$	3. $-x=+3$
2. $+x=-3$	4. $-x=-3$

1 and 4, 2 and 3 are the same equations and therefore $x=\pm 3$ expresses all four equations.

Exercise 84

Solve (correct to .001 where necessary):

1. $x^2 = 12$
2. $x^2 = 75$
3. $x^2 = 55225$
4. $x^2 = 46$
5. $x^2 = \dfrac{169}{1225}$
6. $x^2 = \dfrac{75}{108}$
7. $x^2 = \frac{5}{8}$
8. $x^2 + 10 = 59$
9. $y^2 - 11 = 185$
10. $7m^2 - 175 = 0$
11. $8s^2 - 38 = 90$
12. $11a^2 - 5 = 2 + 2a^2$
13. $3(x-2) - x = 2x(1-x)$
14. $(2t+3)(t+2) - (t+3)(t+4) = 4t^2 - 21$
15. $(t+4)^2 + (t-4)^2 = 48$
16. $\dfrac{3x^2+1}{5} - \dfrac{5(x^2-1)}{10} - \dfrac{(4x^2+1)}{25} = 0$
17. $\dfrac{y^2+y+1}{y-1} - \dfrac{y^2-y+1}{y+1} = 15$
18. $\dfrac{5r-3}{9r+1} = \dfrac{r+2}{2r+5}$
19. $\dfrac{2x-5}{3} = 1.5 - \dfrac{3x+10}{2x+5}$
20. $\dfrac{3x-1}{3x+1} + \dfrac{3x+1}{3x-1} = \dfrac{29}{14}$

21. The length of a rectangle is 3 times its width, and the area is 243 sq. in. Find the dimensions of the rectangle.

22. How long must the side of a square field be that the area of the field may be 5 acres?

23. The dimensions of a rectangle are in the ratio $\frac{2}{3}$, and its area is 300. Find the dimensions of the rectangle.

24. The side of one square is 3 times that of another, and its area is 96 sq. in. more than that of the other. Find the sides of the two squares.

25. If the area of a 3″ circle is 28.2744, find the diameter of a circle whose area is 78.54. (See Exercise 67, problem 3.)

26. Find the diameter of a circular piece of copper whose weight is 3.01 oz. if a 10″ disk weighs 9.03 oz. (See Exercise 67, problem 10.)

27. The intensity of light varies inversely as the square of the distance from the source of light. How far from a lamp should a person sit in order to receive one half as much light as he receives when sitting 3 ft. from the lamp?

28. The distance covered by a falling body varies directly as the square of the time of falling. If a ball drops 402 ft. in 5 seconds, how long will it take it to drop 600 ft.?

29. The weight of an object varies inversely as the square of the distance from the center of the earth. If an object weighs 180 lbs. at the earth's surface, at what distance from the center will it weigh 160 lbs., if the radius of the earth is 4000 miles?

30. The surface of a sphere varies directly as the square of the diameter. Find the diameter of a sphere whose surface is 78.54 sq. in., if the surface of an 11″ sphere is 380.1336 sq. in.

CHAPTER VIII

FORMULAS

Evaluation of Formulas Containing Square Root

Exercise 85

Evaluate the following formulas for the values given (correct to .001 where necessary):

1.	$h=\frac{a}{2}\sqrt{3}$	when $a=5$.
2.	$c=\sqrt{a^2+b^2}$	when $a=4$, $b=5$.
3.	$V=2\,\pi^2r^2R$	when $r=\frac{1}{4}$, $R=1\frac{1}{2}$.
4.	$A=\frac{3}{2}r^2\sqrt{3}$	when $r=3\frac{1}{2}$.
5.	$V=\frac{a^3}{12}\sqrt{2}$	when $a=3.2$.
6.	$G=\sqrt{ab}$	when $a=4$, $b=5$.
7.	$V=\frac{a(\pi r^2+\pi R^2)}{2}+\frac{1}{6}\pi a^3$	when $a=6$, $r=18$, $R=24$.
8.	$t=\pi\sqrt{\frac{l}{g}}$	when $l=1$, $g=32$.
9.	$s=\frac{r}{2}(\sqrt{5}-1)$	when $r=2\frac{1}{2}$.
10.	$D=\sqrt{a^2+b^2+c^2}$	when $a=5$, $b=6$, $c=7$.
11.	$b=\sqrt{a^2+c^2-2a'c}$	when $a=14$, $a'=5$, $c=12$.
12.	$l=2\sqrt{D^2+a^2}+\pi D$	when $D=16$, $a=35$.
13.	$M=\frac{1}{2}\sqrt{2(a^2+c^2)-b^2}$	when $a=15$, $c=17$, $b=19$.
14.	$x=\frac{-b+\sqrt{b^2+4ac}}{2a}$	when $a=3$, $b=5$, $c=20$.

15. $x=\dfrac{-b-\sqrt{b^2+4ac}}{2a}$ when $a=3,\ b=5,\ c=20.$

16. $l=\sqrt{\left(\dfrac{D-d}{2}\right)^2+a^2+\pi\left(\dfrac{D+d}{2}\right)}$ when $D=36, d=6, a=96.$

17. $s=\dfrac{1}{2}r\sqrt{10-2\sqrt{5}}$ when $r=4\frac{1}{5}$.

18. $s=\dfrac{N}{360}.\pi R^2-\dfrac{C}{4}\sqrt{4R^2-C^2}$ when $N=72, R=10, C=13.$

19. $x=\sqrt{r(2r-\sqrt{4r^2-s^2})}$ when $r=3, s=2.$

20. $A=\sqrt{s(s-a)\ (s-b)\ (s-c)}$ when $a=15, b=18, c=22,$ $s=\frac{1}{2}(a+b+c).$

125 A formula is an equation and may be solved for any of the letters involved if the values of all the other letters are given.

Example 1: $Z=4\pi ra$. Find a, when $Z=502.656$, $r=8$.

$$502.656=4\cdot 3.1416\cdot 8\cdot a$$

$$\begin{array}{l}5.\\15.708\\62.832\end{array}$$

$$a=\frac{502.656}{4\cdot 3.1416\cdot 8}=5$$

Example 2: $V=\frac{1}{3}\pi r^2 a$. Find r, when $V=593.7624$, $a=7$.

$$\begin{array}{c}1.0472\\593.7624=\frac{1}{3}\cdot 3.1416\cdot r^2\cdot 7\end{array}$$

$$\begin{array}{l}81\\84.8232\end{array}$$

$$r^2=\frac{593.7624}{1.0472\cdot 7}=81$$

$$r=\pm 9.$$

Example 3: $b^2=a^2+c^2-2ac'$. Solve for c', when $a=5$, $b=6$, $c=7$.

$$36=25+49-2\cdot 5\cdot c'$$
$$10c'=38$$
$$c'=3.8$$

Exercise 86

Find the values (correct to .001 when necessary):

1. $P=4a$ — Find a, when $P=5\frac{1}{2}$.
2. $P=a+b+c$. — Find c, when $P=7962$, $a=1728$, $b=3154$.
3. $P=2(a+b)$. — Find a, when $P=17\frac{1}{8}$, $b=2\frac{1}{4}$.
4. $A=ab$. — Find b, when $A=2.31$, $a=1.1$.
5. $A=\frac{1}{2}bh$. — Find h, when $A=3\frac{17}{20}$, $b=3\frac{1}{2}$.
6. $A=\frac{1}{2}h(b+b')$. — Find h, when $A=96$, $b=18$, $b'=6$.
7. $A=\frac{1}{2}h(b+b')$. — Find b′, when $A=12.8$, $b=1.2$, $h=8$.
8. $C=2\pi r$. — Find r, when $C=50$.
9. $A=\pi r^2$. — Find r, when $A=50$.
10. $w=\frac{l}{h}\cdot p$. — Find p, when $w=333\frac{1}{3}$, $l=25$, $h=4\frac{1}{2}$.
11. $w=\frac{l}{h}\cdot p$. — Find l, when $w=320$, $h=24$, $p=213\frac{1}{3}$.
12. $w=\frac{l}{h}\cdot p$. — Find h, when $w=150$, $l=162$, $p=100$.
13. $L=1\frac{3}{4}d+\frac{1}{8}$. — Find d, when $L=4\frac{1}{16}$.
14. $S=\frac{1}{2}gt^2$. — Find t, when $S=196.98$. (See Exercise 17, problem 5.)
15. $S=\frac{1}{2}gt^2+vt$. — Find v, when $S=164.72$, $t=3$.

16. $F=\dfrac{uv}{u+v}$. Find F, when $u=11$, $v=7$.

17. $F=\dfrac{uv}{u+v}$. Find v, when $F=1\frac{7}{8}$, $u=3$.

18. $x=\dfrac{-b-\sqrt{b^2+4ac}}{2a}$. Find x, when $b=-5$, $a=3$, $c=-2$.

19. $A=\dfrac{\pi D^2}{4}$. Find D, when $A=115$.

20. $V=\pi r^2 a$. Find r, if $V=330$, $a=7$.

21. $V=\pi r^2 a$. Find a, if $V=46.9$, $r=2.3$.

22. $V=\pi r^2 . \dfrac{h+H}{2}$. Find r, when $V=1932$, $H=14.6$, $h=8.2$.

23. $V=\pi r^2 . \dfrac{h+H}{2}$. Find H, when $V=2246$, $r=8$, $h=6$.

24. $A=\dfrac{abc}{4r}$. Find A, when $a=2.3$, $b=3.2$, $c=4.1$, $r=2.058$.

25. $A=\dfrac{abc}{4r}$. Find r, when $a=21$, $b=28$, $c=35$, $A=294$.

26. $A=\frac{1}{2}r(a+b+c)$. Find r, when $a=79.3$, $b=94.2$, $c=66.9$, $A=261.012$.

27. $A=\frac{1}{2}r(a+b+c)$. Find a, when $A=27.714$, $r=2.3095$, $b=8$, $c=8$.

28. $A=\frac{1}{2}(2\pi R+2\pi r)s$. Find A, when $R=8$, $r=3$, $s=7$.

29. $A=\frac{1}{2}(2\pi R+2\pi r)s$. Find r, when $A=439.824$, $R=10$, $s=10$.

30. $A=\frac{1}{2}(2\pi R+2\pi r)s$. Find s, when $A=106.029$, $R=7\frac{1}{2}$, $r=6$.

31. $l=2\sqrt{\left(\dfrac{D+d}{2}\right)^2+a^2}+\pi\left(\dfrac{D+d}{2}\right)$.
Find l, when $D=1\frac{1}{2}$, $d=1\frac{1}{4}$, $a=15$.

32. $h=\dfrac{a}{2}\sqrt{3}$. Find a, when $h=27.7136$.

33. $A=\frac{h^2}{3}\sqrt{3}$. Find h, when $A=\frac{10}{3}\sqrt{3}$.

34. $A=\frac{a^2}{4}\sqrt{3}$. Find a, when $A=\sqrt{48}$.

35. $V=\frac{a^3}{12}\sqrt{2}$. Find V, when $a=6$.

36. $A=\frac{3r^2}{2}\sqrt{3}$. Find r, when $A=153$.

37. $c^2=a^2+b^2$. Find b, when $c=2.1$, $a=1.7$.

38. $b^2=a^2+c^2+2a'c$. Find a, when $b=8$, $c=5$, $a'=2.1$.

39. $b^2=a^2+c^2+2a'c$. Find a', when $a=18$, $b=16$, $c=31$.

40. $b^2=a^2+c^2-2ac'$. Find c, when $a=5$, $b=4$, $c'=2.3$.

41. $b^2=a^2+c^2-2ac'$. Find c', when $a=14$, $b=15$, $c=16$.

42. $H=\frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$.
Find H, when $a=2.18$, $b=5$, $c=3.24$, $s=\frac{1}{2}(a+b+c)$.

43. $a^2+b^2=2\left(\frac{c}{2}\right)^2+2m^2$. Find m, when $a=9$, $b=12$, $c=15$.

44. $a^2+b^2=2\left(\frac{c}{2}\right)^2+2m^2$. Find b, when $a=5$, $c=13$, $m=6\frac{1}{2}$.

45. $s=\frac{r}{2}(\sqrt{5}-1)$. Find r, when $s=10.50685$.

46. $x=\frac{-b+\sqrt{b^2+4ac}}{2a}$. Find x, when $a=3$, $b=-7$, $c=+2$.

47. $V=2\pi^2r^2R$. Find r, when $V=98696.5056$, $R=50$.

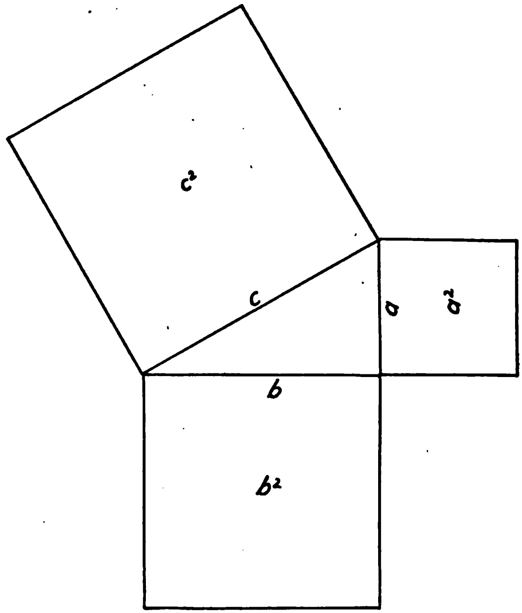
48. $x=\sqrt{r(2r-\sqrt{4r^2-s^2})}$. Find x, when $r=s=10$.

49. $s=\frac{N}{360}\cdot\pi r^2-\frac{c}{4}\sqrt{4r^2-c^2}$.
Find N, when $s=23.1872$, $c=r=16$.

50. $x=\frac{-b-\sqrt{b^2+4ac}}{2a}$. Find x, when $a=-6$, $b=-9$, $c=+2$.

Right Triangle

126 One of the formulas most commonly used is that of the right triangle.



$a^2 + b^2 = c^2$

Fig. 70

127 Right Triangle: A *right triangle* is a triangle in which one angle is a right angle. The lines including the right angle are called the *sides*, and the line opposite the right angle is called the *hypotenuse*.

It can be proved that;

128 The square of the hypotenuse is equal to the sum of the squares of the two sides.

This truth is stated by the formula:

$$c^2 = a^2 + b^2 \text{ (Fig. 70).}$$

Exercise 87

Find results correct to .001 when necessary:

1. Find c, when $a=8$, $b=15$.

2. Find a, when $b=9$, $c=41$.

3. Find b, when $a=3$, $c=6$.

4. Find the hypotenuse of a right triangle when the sides are 3.2 and 2.4.

5. The hypotenuse and one side of a right triangle are respectively $2\frac{3}{4}$ and $1\frac{7}{8}$. Find the other side.

6. The sides of a right triangle are $5\frac{1}{4}$ and 12.5. Find the hypotenuse.

7. The two sides of a right triangle are equal to each other, and the hypotenuse is 18. Find the sides. (Fig. 71.)

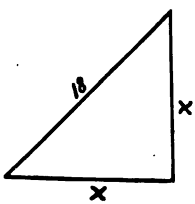


Fig. 71

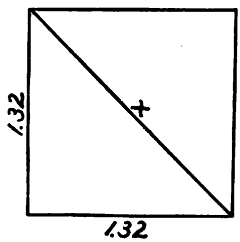


Fig. 72

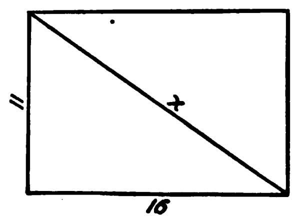


Fig. 73

8. One side of a right triangle is 3 times the other, and the hypotenuse is 80. Find the sides. Draw a figure.

9. The two sides of a right triangle are in the ratio $\frac{3}{4}$, and the hypotenuse is 225. Find the sides. Draw a figure.

10. Find the diagonal of a square whose sides are 1.32. (Fig. 72.)

11. Find the perimeter of a square whose diagonal is 17. Draw a figure.

12. Find the diagonal of a rectangle whose dimensions are 11 and 16. (Fig. 73.)

13. Find the dimensions of a rectangle whose diagonal is 91, if the length is 5 times the width. Draw a figure.

14. The perimeter of a rectangle is 70, and its sides are in the ratio $\frac{3}{7}$. Find the diagonal.

15. A ladder 36 ft. long is placed with its foot 11 ft. from the base of the building. How high is a window which the ladder just reaches?

16. A flag staff 79 ft. long is broken 29 ft. from the ground. If the parts hold together, how far from the foot of the staff will the top touch the ground?

17. How long is a guy wire which is attached to a wireless tower 227 ft. from the ground, and is anchored 362 ft. from the foot of the tower?

18. The slant height of a cone is 12″, and the radius of the base is $5\frac{1}{2}$″. Find the altitude of the cone. (Fig. 74.)

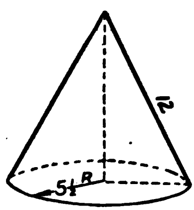


Fig. 74

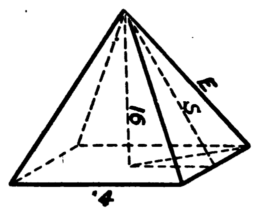


Fig. 75

19. One side of the base of a square pyramid is 14″, and the altitude is 16″. Find the edge, E. (Fig. 75.) (Suggestion: The altitude of the pyramid meets the base at the middle point of the diagonal.)

20. Find the slant height, S. (Fig. 75.)

CHAPTER IX

QUADRATIC EQUATIONS

129 The quadratic equations in Chapter VII, Art. 124, were solved in the same way as equations containing only the first power of the unknown, until they were reduced to the form $x^2=20$. To continue from this point, it became necessary to extract the square root of both members of the equation in order to find x. Thus,

$$x^2=20$$

$$x=\pm 4.4722$$

130 Quadratic equations when simplified *may contain both the first and second powers of the unknown.* For example: $x^2+6x=72$. It is not possible to extract the square root of x^2+6x, and therefore the form of the equation must be changed before extracting the square root of both members. By Art. 115, x^2+6x+9 is a trinomial square, because x^2 and 9 are squares and 6x is 2 times the product of their square roots. Therefore by adding 9 to both members of $x^2+6x=72$, it becomes $x^2+6x+9=81$, and then it is possible to extract the square root of both members and $x+3=\pm 9$. (See Exercise 76.)

131 This example shows the necessity of finding the third term of a trinomial square when the first two are given. This operation is called *completing the square.*

Let it be required to "complete the square" in the expression x^2+10x. In the trinomial square, 10x will be 2 times the product of the square root of x^2 and the square root of the term which is required. Since $\sqrt{x^2}=x$, then 10 must be 2 times the square root of the required term, 5 must be the square root of the required term, and 25 must be the required term. This may be expressed as a problem thus:

Let k^2 = the required 3d term.

$x^2+10x+k^2$ = the trinomial square

then $10x=2\cdot\sqrt{x^2}\cdot\sqrt{k^2}$

$$10x=2\cdot x\cdot k$$

$$k=\frac{10x}{2x}=\frac{10}{2}$$

$k^2=\left(\frac{10}{2}\right)^2$ (the required third term)

132 **RULE: To change an expression of the form x^2+ax into a trinomial square, add to it the square of $\frac{1}{2}$ the coefficient of x.**

Exercise 88

Complete the square in the following:

1. x^2+4x	6. x^2-17x
2. x^2+12x	7. $x^2-\frac{1}{2}x$
3. x^2-6x	8. x^2+x
4. x^2+5x	9. $x^2+3\frac{1}{2}x$
5. x^2+9x	10. $x^2+3.2x$

Solution of Quadratic Equations by Completing the Square

133 Example 1: Solve $x^2+5x=14$.

$$x^2+5x=14$$

$x^2+5x+\frac{25}{4}=\frac{81}{4}$ (adding $\frac{25}{4}$ to both members. See Art. 132.)

$x+\frac{5}{2}=\pm\frac{9}{2}$ Why?

$$x+\frac{5}{2}=+\frac{9}{2} \qquad \text{or} \qquad x+\frac{5}{2}=-\frac{9}{2}$$

$$x=+2 \qquad\qquad x=-7$$

$$\therefore x=+2,-7.$$

Check: $(+2)^2+5(+2)=14 \qquad (-7)^2+5(-7)=14$

$$4+10=14 \qquad \text{or} \qquad 49-35=14$$

$$14=14 \qquad\qquad 14=14$$

Example 2: Solve $x^2-3x-6=0$.

$$x^2-3x-6=0$$

$$x^2-3x \quad =6 \qquad \text{Why?}$$

$$x^2-3x+\tfrac{9}{4}=\tfrac{33}{4} \qquad \text{Why?}$$

$$x-\tfrac{3}{2}=\pm\frac{\sqrt{33}}{2} \qquad \text{Why?}$$

$$x=\tfrac{3}{2}\pm\frac{\sqrt{33}}{2} \qquad \text{Why?}$$

$$x=\tfrac{3}{2}\pm\frac{5.745}{2} \qquad \text{Why?}$$

$$x=4.373, \text{ or } -1.373$$

Check: Results which contain the square roots of numbers which are not *perfect* squares, will not check in the problem *perfectly*.

$(4.373)^2-3(4.373)-6=0 \qquad (-1.373)^2-3(-1.373)-6=0$

$19.123129-13.119-6=0 \quad \text{or} \quad 1.885129+4.119-6=0$

$$.004129=0 \qquad\qquad .004129=0$$

An approximate check.

Example 3: Solve $\frac{7}{r^2}-5=-\frac{2}{r}$.

$$\frac{7}{r^2}-5=-\frac{2}{r}$$

$$7-5r^2=-2r \quad \text{Why?}$$

$$5r^2-7=2r \quad \text{(multiplying both members by}-1\text{).}$$

$$5r^2-2r=7 \quad \text{Why?}$$

$$r^2-\frac{2r}{5}=\frac{7}{5} \quad \text{(dividing both members by 5).}$$

$$r^2-\frac{2r}{5}+\frac{1}{25}=\frac{36}{25} \quad \text{Why?}$$

$$r-\frac{1}{5}=\pm\frac{6}{5} \quad \text{Why?}$$

$$r=\frac{1}{5}\pm\frac{6}{5}$$

$$r=\frac{7}{5}, \text{ or } -1$$

Observe:

I. The signs of an equation may be changed by multiplying (or dividing) both members by a *negative* number.

II. The terms containing the unknown should be arranged in the first member and the known term in the second member.

III. If the term containing the square of the unknown has a coefficient other than 1, it is necessary to divide both members of the equation by that coefficient before completing the square according to Art. 132.

Exercise 89

Solve: (Express the results correct to .001 when necessary.)

1. $x^2+4x=5$
2. $x^2-12x=45$
3. $y^2=4y+12$
4. $x^2=91-6x$
5. $x^2+3=4x$
6. $x^2-2x-15=0$
7. $x^2-7x+10=0$
8. $x^2-6=x$
9. $x^2-12x+6=\frac{1}{4}$
10. $x^2-3x+2=12$
11. $x^2+4=5x$
12. $x^2+48=20x-3$
13. $x^2-4x=8$
14. $x^2+1=6x$
15. $x^2+6x+6=0$
16. $x^2=23+4x$
17. $8y+21=y^2$
18. $t^2+3t=9$
19. $s^2+11=7s$
20. $m^2+5m=-2$
21. $2x^2-5x=3$
22. $3x^2+5x=2$
23. $5x^2-7=3x-5$
24. $12(x^2+1)=25x$
25. $2(x^2+1)=5(x+1)$
26. $7x^2-4x=18$
27. $3x^2-14x+6=0$
28. $5x^2-12=11x$
29. $2x^2-3x=18$
30. $3x^2-17x=-16$

Exercise 90

Solve and check:

1. $\frac{2}{3}x^2-4\frac{2}{3}x=-2\frac{1}{6}$
2. $\frac{3}{5}x^2+18+1\frac{1}{5}x=\frac{9}{5}x-3x^2+39$
3. $\frac{x}{3}+\frac{3}{x}-\frac{x}{4}+1$
4. $\frac{1}{9x^2}-\frac{11}{18x}=-\frac{5}{6}$
5. $(3x-5)(4x+2)-2x(x+7)-81=x+14$

6. $3x+\frac{24}{x-1}=5x-4$

7. $(2x-1)^2-(3x+2)^2+(3x-1)^2+30=0$

8. $\frac{8x}{x+2}-\frac{20}{3x}=6$

9. $\frac{6y}{y+1}+\frac{6(y+1)}{y}=13$

10. $\frac{x}{x+84}=\frac{5}{3x-7}$

Solve:

11. $\frac{x-6}{3}=\frac{-2\frac{1}{3}}{2-x}$

12. $\frac{2x+3}{5x-1}=\frac{x-2}{3x-2}$

13. $(5x-6)^2=(11+3x)(7-8x)+7x$

14. $(x-2)(x-3)(2x+1)-(x+7)(2x-7)(x+3)=65(2+x)$
$-8(x^2-5)$

15. $\frac{x-1}{x}+\frac{x}{x+1}=\frac{5}{3}$

16. $\frac{5}{x+2}=\frac{14}{x-4}-\frac{3}{x}$

17. $\frac{15+3x}{x+1}-7=\frac{24}{x+1}-\frac{30+4x}{x+3}$

18. $\frac{3x+2}{3x-2}+\frac{3x-2}{3x+2}=\frac{15x+10}{3x+2}$

19. $\frac{8-x}{2}-\frac{2x-11}{x-3}=\frac{x-1}{3}$

20. $\frac{x}{2x-1}=\frac{13}{3}-\frac{8+x}{x+5}$

Exercise 91

1. The length of a rectangle is 5 more than the width, and the area is 104. Find the dimensions.

2. The product of two consecutive numbers is 272. Find them.

3. The sum of the squares of two consecutive numbers is 421. Find the numbers.

4. The difference of the squares of two consecutive numbers is 83. Find the numbers.

5. The hypotenuse of a right triangle is 13, and one side is 7 more than the other. Find the sides.

6. The diagonal of a rectangle is 41, and the length is 31 more than the width. Find the dimensions.

7. The sum of the squares of three consecutive numbers is 194. Find the numbers.

8. The sum of the area of two squares is 202, and a side of one is 2 more than a side of the other. Find the sides of both squares.

9. If the product of three consecutive numbers is divided by each of them in turn, the sum of the three fractions obtained is 242. Find the numbers.

10. The length of a rectangle is 4 more than 3 times the width, and the area is 64. Find the dimensions.

11. Find three consecutive numbers such that the product of the first two plus the product of the second and third shall equal 882.

12. Divide 17 into two parts whose product shall be 60. (Suggestion: Let $x =$ one part, $17 - x =$ the other part.)

13. The square of the larger of two consecutive numbers is equal to 122 more than twice the smaller number. Find the numbers.

14. The side of one square is $\frac{3}{5}$ that of another, and the sum of their perimeters is 96. Find the sum of their areas.

15. Divide 22 into two parts such that one is the square of the other.

16. The perimeter of a rectangle is 60, and its area is 56. Find the dimensions.

17. The differences between the hypotenuse of a right triangle and the two sides, are 8 and 4. Find the hypotenuse.

18. The perimeter of a rectangle is 34″, and the area is 109 sq. in. less than the square of the diagonal. Find the dimensions.

19. The length of one rectangular field is 3 times the width. Another field which is 4 rds. longer and 3 rds. wider, contains 44 sq. rds. less than twice the area of the first. Find the dimensions of both fields.

20. If 12 is divided by each of two consecutive numbers, the sum of the results is $\frac{17}{6}$. Find the consecutive numbers.

21. The ratio of the squares of two consecutive even numbers is $\frac{81}{100}$. Find the even numbers. (Hint: How far apart are two consecutive even numbers?)

22. The area of two circles are in the ratio $\frac{3}{7}$, and the diameter of the larger is 2″ more than the diameter of the smaller. Find the diameter of each. (See Exercise 67, problem 3.)

23. One of two pulleys connected by a belt has a circumference 18″ larger than the other. The R. P. M. of one is 36 more than the R. P. M. of the other. If the belt travels 2016 ft. per min., find the two circumferences. (See Example 2, Art. 105.)

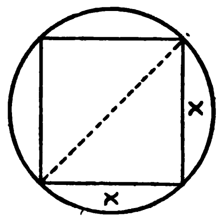


Fig. 76

24. The area of a circle is 78.54. Find the side of the inscribed square. (Fig. 76.)

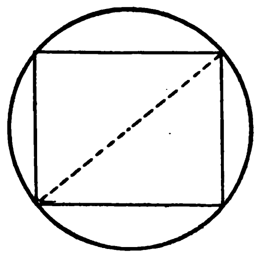


Fig. 77

25. The area of a circle is 314.16. The sides of the inscribed rectangle are in the ratio $\frac{3}{4}$. Find the sides of the rectangle. (Fig. 77.)

CHAPTER X

EQUATIONS AND FACTORING

134 Factors: The *factors* of a number are the numbers whose product is equal to the given number. For example: 1 and 12, 2 and 6, 3 and 4, 2, 3 and 2 are all sets of factors of 12 because in each case their product is equal to 12; $5ac^2$ and $6a^2b$ form one set of factors of $30a^3bc^2$ because $5ac^2 \cdot 6a^2b = 30a^3bc^2$; $(x-2)$, $(x-3)$, and x form one set of factors of x^3-5x^2+6x because $(x-2)(x-3)x = x^3-5x^2+6x$.

135 Factoring: *Factoring* is the process of separating a number into its factors.

136 Any equation that can be put into its factored form can be easily solved. Let it be required to solve the equation $(x-2)(x-1)(x+4)=0$. It is readily seen that if $x=+2$, the equation will check because $(2-2)(2-1)(2+4)=0 \cdot 1 \cdot 6=0$; also if $x=+1$, the equation will check because $(1-2)(1-1)(1+4)=(-1) \cdot 0 \cdot 5=0$; also if $x=-4$, the equation will check because $(-4-2)(-4-1)(-4+4)=(-6)(-5) \cdot (0)=0$. In this equation then, $x=+2$, $+1$, or -4. These results are obtained by putting each factor equal to zero and solving the equations thus formed as follows:

$$x-2=0 \qquad x-1=0 \qquad x+4=0$$

$$\text{or} \qquad\qquad \text{or}$$

$$x=2 \qquad x=1 \qquad x=-4$$

Observe that the *second* member of the equation must be *zero* in order to check when one of the factors equals zero.

Exercise 92

Solve and check:

1. $(x+1)(x-2)=0$
2. $(x-1)(x+2)=0$
3. $(x-3)(x-5)=0$
4. $(x+7)(x+4)=0$
5. $x(x-6)=0$
6. $(6-x)x=0$
7. $(2x-1)(3x+5)=0$
8. $(4x+2)(x^2-1)(2x-1)=0$
9. $(x+\frac{7}{8})(x-\frac{5}{6})(x+\frac{2}{3})(x-\frac{4}{5})=0$
10. $(3x-\frac{1}{2})(x^2+4x+3)(5x-.2)(\frac{2}{3}x+6)=0$

137 In order to express an equation in its factored form, a knowledge of factoring is necessary. Factoring is the reverse of multiplication, and therefore a study of some of the common forms of multiplication is necessary before a study of factoring.

Multiplication of any two Binomials

138 Find the value of the following by multiplying and write the results as in part 1:

1. $(x+2)(x+5)=x^2+7x+10.$

$$\begin{array}{l}
x+2 \\
\quad\times \\
\underline{x+5} \\
x^2+2x \\
\underline{\quad\ +5x+10} \\
x^2+7x+10
\end{array}$$

2. $(x+3)(x-1)=$
3. $(x-4)(x-1)=$
4. $(-x+5)(x+7)=$
5. $(2x+1)(3x+4)=$
6. $(3x-5)(2x-3)=$
7. $(4x+1)(5x-2)=$
8. $(5x+7)(3x-2)=$

In each result, observe the following:

I. The first term of the result is the product of the first terms of the binomials.

II. The last term of the result is the product of the second terms of the binomials.

III. The second term of the result is the sum of the two cross products. (See part 1.)

139 **RULE: To multiply any two binomials, find the product of the two first terms of the binomials, the sum of the two cross products, the product of the two second terms of the binomials, and write the result as a trinomial.**

Exercise 93

Write the results without written multiplication:

1.	$(x+2)(x+3)$	7.	$(2x-5)(2x+7)$
2.	$(y-7)(y-11)$	8.	$(7x+2)(7x+2)$
3.	$(t+3)(t-5)$	9.	$(4x-5z)(4x-5z)$
4.	$(x+8)(x-1)$	10.	$(3x-8)(3x+8)$
5.	$(2x+y)(x+5y)$	11.	$(3x^2-4y^2)(4x^2-3y^2)$
6.	$(x-3)(5x+7)$	12.	$(a^3+2ab^2)(5a^3-7ab^2)$

Factoring a Trinomial of the form ax^2+bx+c.

140 The process of multiplying any two binomials may be reversed, and a trinomial of the form ax^2+bx+c may be factored into two binomials.

Example 1: Factor x^2+x-6.

$$x^2+x-6=(x+3)(x-2) \text{ because } \left\{ \begin{array}{l} x^2=x\cdot x \\ -6=(+3)(-2) \\ +x=(+3)(x)+(-2)(x) \\ \quad \text{(See Art. 138)} \end{array} \right.$$

Example 2: Factor $14x^2-3x-2$.

As far as the first and last terms of the trinomial are concerned, the factors may be any of the following:

$$\begin{array}{cccc} 7x+1 & 7x-1 & 7x-2 & 7x+2 \\ \underline{2x-2} & \underline{2x+2} & \underline{2x+1} & \underline{2x-1} \\ & & & \\ 14x+1 & 14x-1 & 14x+2 & 14x-2 \\ \underline{x-2} & \underline{x+2} & \underline{x-1} & \underline{x+1} \end{array}$$

But the only set which gives the correct middle term of the trinomial is $(7x+2)(2x-1)$ $\therefore 14x^2-3x-2=(7x+2)(2x-1)$.

Exercise 94

Factor:

1. $x^2+7x+10$
2. $x^2-7x+10$
3. x^2+4x+4
4. x^2+3x+2
5. x^2-4x+3
6. $x^2-2x-15$
7. $x^2+2x-15$
8. $x^2-21x+110$
9. $x^2-x-110$
10. $x^2+x-110$
11. $5m^2-2m-7$
12. $2y^2-3y-5$
13. $7x^2-2x-5$
14. $2x^2+8x+6$
15. $3x^2-10x+8$
16. $3x^2-10x-8$
17. $3y^2+5y-12$
18. $2x^2-7x+6$
19. $8y^2-26y-7$
20. $2d^2+17dx-30x^2$
21. $15-18x-24x^2$
22. $35a^4+18a^2-8$
23. $26ab-15a^2+21b^2$
24. $15m^2+19mn-56n^2$
25. $15m^2+19mn+6n^2$

Solution of Quadratic Equations by Factoring

141 Example: Solve $10x^2-3x=18$ by factoring.

$$10x^2-3x=18$$

$$10x^2-3x-18=0 \quad \text{Why?}$$

$$(5x+6)(2x-3)=0 \quad \text{Why?}$$

$$5x+6=0 \quad 2x-3=0$$

or

$$x=-\tfrac{6}{5} \quad x=\tfrac{3}{2}.$$

Check: $10(-\frac{6}{5})^2-3(-\frac{6}{5})=18$ $\quad 10(\frac{3}{2})^2-3(\frac{3}{2})=18$

$10(\frac{36}{25})+\frac{18}{5}=18$ $\quad$ or $10(\frac{9}{4})-3(\frac{3}{2})=18$

$\frac{72}{5}+\frac{18}{5}=18$ $\quad \frac{45}{2}-\frac{9}{2}=18$

$18=18$ $\quad 18=18$

NOTE: If a quadratic equation can not be factored readily, it can always be solved by completing the square.

Exercise 95

Solve by factoring and check:

1. $x^2+7x+12=0$
2. $x^2-17x+70=0$
3. $x^2-16x=36$
4. $x^2+22x=75$
5. $x^2-15x-33=1$
6. $x^2-6x+30=6x-6$
7. $2x^2+5x+2=0$
8. $3x^2+20x=7$
9. $5x^2-6=13x$
10. $x+3=14x^2$
11. $6(x^2-1)=5x$
12. $12x^2-29x+16=1$
13. $2x(12x-1)=15$
14. $12x(x-20)=x-20$
15. $49x^2+8=42x$
16. $16a^2+40a=-25$
17. $48t^2=22t+5$
18. $5m(10m-11)+14=0$
19. $27s^2-42s+16=0$
20. $72y^2+6y-15=0$

Exercise 96. Factoring Applied to Fractions

Reduce to lowest terms:

1. $\dfrac{x^2-10x+9}{x^2+5x-6}$

2. $\dfrac{6x^2-7bx-3b^2}{2x^2-11bx+12b^2}$

3. $\dfrac{12x^2-xy-6y^2}{6x^2+11xy+3y^2}$

4. $\dfrac{10x^2-21xy-10y^2}{6x^2-xy-35y^2}$

5. $\dfrac{3y^2-57y+54}{15y^2-36y+21}$

Simplify:

6. $\dfrac{y^2-4y-21}{4y^2+12y+9}\cdot\dfrac{2y^2-5y-12}{y^2-11y+28}$

7. $\dfrac{x+3}{2x^2-3x-2}\cdot\dfrac{6x^2+7x+2}{4x^2+7x-15}$

8. $\dfrac{6x^2-5xy-6y^2}{8x^2-14xy+3y^2}\cdot\dfrac{20x^2+19xy-6y^2}{15x^2+28xy+12y^2}$

9. $\dfrac{1}{x^2-5x+6}\div\dfrac{1}{2x^2+3x-9}$

10. $\dfrac{12x-1}{4x-3}\cdot\dfrac{20x^2-23x+6}{12x^2-145x+12}\div\dfrac{20x^2-23x+6}{4x^2-51x+36}$

142 In problems involving the addition or subtraction of fractions, it is always advisable, if possible, to factor the denominators in order to find the *lowest* common denominator.

Example: Solve $\dfrac{36}{6x^2-17x+5}-\dfrac{9}{2x^2+x-15}=\dfrac{20}{3x^2+8x-3}$

$$\frac{36}{6x^2-17x+5}-\frac{9}{2x^2+x-15}=\frac{20}{3x^2+8x-3}$$

$$\frac{36}{(2x-5)(3x-1)}-\frac{9}{(2x-5)(x+3)}=\frac{20}{(x+3)(3x-1)}\quad\text{Why?}$$

$$36(x+3)-9(3x-1)=20(2x-5)\quad\left\{\begin{matrix}\text{L. C. D. is }(2x-5)\\(x+3)(3x-1).\end{matrix}\right\}$$

$$36x+108-27x+9=40x-100 \quad \text{Why?}$$

$$217=31x \quad \text{Why?}$$

$$x=7 \quad \text{Why?}$$

Check: $\dfrac{36}{294-119+5}-\dfrac{9}{98+7-15}=\dfrac{20}{147+56-3}$

$$\frac{36}{180}-\frac{9}{90}=\frac{20}{200}$$

$$\frac{1}{5}-\frac{1}{10}=\frac{1}{10}$$

$$\frac{1}{10}=\frac{1}{10}$$

Exercise 97

Solve:

1. $\dfrac{1}{x-3}+\dfrac{1}{x-2}=\dfrac{11}{x^2-5x+6}$

2. $\dfrac{9}{5(x-2)}+\dfrac{24}{x^2-3x+2}=2\frac{3}{5}$

3. $\dfrac{4}{3x-2}=\dfrac{3}{2x-3}-\dfrac{10}{6x^2-13x+6}$

4. $\dfrac{x+1}{3(x-2)}-\dfrac{x-1}{2(x+3)}=\dfrac{x+4}{x^2+x-6}$

5. $\dfrac{3x-2}{2x^2-13x+20}-\dfrac{2x-5}{3x^2-14x+8}=\dfrac{x-4}{6x^2-19x+10}$

143 Example:

Simplify: $\frac{2x-1}{x^2-2x-3}-\frac{5(x-3)}{2x^2-5x-3}+\frac{x+3}{2x^2+3x+1}$

$$\frac{2x-1}{x^2-2x-3}-\frac{5(x-3)}{2x^2-5x-3}+\frac{x+3}{2x^2+3x+1}$$

$$=\frac{2x-1}{(x-3)(x+1)}-\frac{5(x-3)}{(2x+1)(x-3)}+\frac{x+3}{(2x+1)(x+1)} \quad \text{Why?}$$

$$=\frac{(2x-1)(2x+1)}{(x-3)(x+1)(2x+1)}-\frac{5(x-3)(x+1)}{(2x+1)(x-3)(x+1)}$$

$$+\frac{(x+3)(x-3)}{(2x+1)(x+1)(x-3)} \quad \text{(Reducing to L. C. D.)}$$

$$=\frac{4x^2-1-5x^2+10x+15+x^2-9}{(x-3)(x+1)(2x+1)}$$

$$=\frac{10x+5}{(x-3)(x+1)(2x+1)}=\frac{5(2x+1)}{(x-3)(x+1)(2x+1)}=\frac{5}{(x-3)(x+1)}$$

Observe that in addition and subtraction of fractions, the L. C. D. is found in the same way as for clearing of fractions in equations, but the L. C. D. *must be retained* in the result.

144 The process of reducing mixed expressions to a fractional form becomes a special case of addition and subtraction of fractions, if an integral number is considered a fraction with the denominator 1.

Example: Reduce $a+\frac{b}{a+b}$ to fractional form.

$$a+\frac{b}{a+b}=\frac{a(a+b)}{a+b}+\frac{b}{a+b}$$

$$=\frac{a^2+ab+b}{a+b}$$

Exercise 98

Simplify:

1. $\dfrac{7}{x^2+6x-16}-\dfrac{4}{x^2+7x-18}$

2. $\dfrac{3-x}{x^2-9x+20}+\dfrac{x+4}{7x^2-26x-8}$

3. $\dfrac{x}{5a^2-4a-12}-\dfrac{y}{a^2+4a-12}+\dfrac{z}{a-2}$

4. $\dfrac{x-1}{5x^2-17x+14}-\dfrac{2x}{10x^2-19x+7}$

5. $\dfrac{a}{a^2+3ab+2b^2}+\dfrac{b}{a^2-ab-2b^2}-1$

145 Complex Fraction. A *complex fraction* is one in which the numerator, or denominator, or both contain fractions.

146 **RULE: To simplify a complex fraction, simplify both numerator and denominator (if necessary) and divide the numerator by the denominator.**

Simplify:

6. $\dfrac{1+\dfrac{1}{a}}{1-\dfrac{1}{a}}$

7. $\dfrac{\dfrac{(a+b)^2}{4ab}-1}{\dfrac{(a-b)^2}{4ab}+1}$

8. $\dfrac{\dfrac{1}{a+8}-\dfrac{1}{a-7}}{\dfrac{1}{a+8}-\dfrac{1}{a-4}}$

9. $\dfrac{\dfrac{6x^2-x-2}{x+1}-6x}{\dfrac{6x^2+7x+2}{x^2+2x+1}-\dfrac{6x}{x+1}}$

10. $\dfrac{\dfrac{3a^2-2a-1}{3a^2+a-1}\cdot\dfrac{2a^2+5a-3}{3a^2+7a+2}}{\dfrac{4a^2-2a-2}{4a^2+10a+4}}$

Factoring a Trinomial Square

147 By Art. 111, a trinomial square is obtained by squaring a binomial (multiplying a binomial by itself), which makes it a special case of Art. 138.

Finding the square root of a trinomial square (Art. 117) is finding one of its two equal factors, and, therefore, a trinomial square may be factored by expressing it as the square of its binomial square root. For example: $9x^2-12xy+4y^2=(3x-2y)^2=(3x-2y)(3x-2y)$.

Exercise 99

Factor:

1. $t^2-16t+64$
2. $100+s^2+20s$
3. $81+225x^2-270x$
4. $81m^2-216m+144$
5. $-30rs^2+9r^2+25s^4$
6. $121-44x^6+4x^{12}$
7. $64m^4+176m^2n+121n^2$
8. $169r^2s^4-78rs^2v^3+9v^6$
9. $\frac{4x^2}{25}-\frac{8}{5}ax+4a^2$
10. $\frac{9}{16}t^2-\frac{9}{7}t+\frac{36}{49}$

Factoring the Difference of Two Squares

148 By Art. 138, if the two binomials to be multiplied together are the sum and difference of the same two terms, the sum of the cross products is zero, and the result becomes the square of the first term of the binomials minus the square of the second term of the binomials. For example: $(2x+5)(2x-5)=4x^2-25$.

149 By reversing the process, the difference of two squares may be factored into two binomials, one the sum, and the other the difference, of their square roots.

Exercise 100

Factor:

1. x^2-y^2
2. b^2-a^4
3. $16-m^2$
4. $1-a^2$
5. $t^2-\frac{1}{4}$
6. $144x^2b^4-9$
7. $\frac{4}{9}m^2-\frac{25}{16}n^2$
8. a^4-b^4 (into 3 factors)
9. $a^2b^2-9m^2n^2$
10. $16a^4-1$ (into 3 factors)
11. $1296x^4-625y^4$ (into 3 factors)
12. $a^{10}-b^{10}$
13. $m^{16}-n^{16}$ (into 5 factors)
14. $\frac{25}{x^2}-\frac{a^2}{49}$
15. $\frac{1}{m^2}-\frac{4n^2}{x^2}$

Exercise 101

Solve:

1. $\frac{3x-1}{x-2}=\frac{13x+9}{x^2-4}$
2. $\frac{3}{5}=\frac{x+3}{x-3}-\frac{4}{x^2-9}$
3. $\frac{1}{2x-5}-\frac{1}{2x+5}=\frac{x}{4x^2-25}$
4. $\frac{2}{3}-\frac{3}{2x+1}+\frac{4}{2x-1}=\frac{11}{15}+\frac{x}{4x^2-1}$
5. $\frac{2x-1}{x^2+14x+49}=\frac{1}{x^2-49}$
6. $\frac{x-1}{4x^2-20x+25}=\frac{2x+5}{2x^2-3x-5}$
7. $\frac{2(y+1)}{4y^2-9}=\frac{2y+5}{4y^2+12y+9}$
8. $\frac{4}{x-1}-\frac{12}{x+1}=\frac{9}{x^2-1}-\frac{3}{x^2-2x+1}$
9. $\left(x+1-\frac{30}{x}\right)\cdot\frac{x}{x^2-10x+25}=\frac{12x^2-11x-15}{4x^2+17x+18}$
10. $\dfrac{\frac{1-x^2}{1+x^2}-\frac{1+x^2}{1-x^2}}{\frac{1-x}{1+x}-\frac{1+x}{1-x}}=\frac{1}{4}$

CHAPTER XI

LITERAL EQUATIONS

150 Literal Equations. Sometimes the solution of an equation results in a *formula* containing one or more *general numbers.* Such equations are called *literal equations.*

Example. Solve: $3x+15b=27b-x$

$$4x = 12b$$
$$x = 3b$$

Check: $3\cdot 3b+15b=27b-3b$

$$24b=24b$$

This solution means that the value of x is always three times the value of b.

151 In Exercises 53, 54, and 55, literal equations of the following forms were solved:

1. $3a^2b^3x=-12a^3b^3$
2. $3a^2m^3x=1.11\ a^3m^3-3.3\ a^2m^4$
3. $(y+2)x=y^3-y^2-34y-56.$

These equations were solved in each case by dividing both members of the equation by the coefficient of the unknown.

To reduce an equation to any one of these forms, use the following steps:

1. Clear of fractions.
2. Combine similar terms.
3. Transform the equation so that *all the terms containing the unknown* stand in *one* member and *all other terms* in the *other* member.
4. Divide both members of the equation by the coefficient of the unknown.

152 In some cases, after applying three of these steps, it will not be possible to combine the terms containing the unknown quantity. In those cases, it is necessary to express that member as the product of the unknown and a polynomial in order to find the coefficient of the unknown.

Example: Solve for x: $\frac{x-a}{x-b}=\frac{b}{a}$

$ax-a^2=bx-b^2$ Why?

$ax-bx=a^2-b^2$ Why?

$x(a-b)=a^2-b^2$ Why?

$$x=\frac{a^2-b^2}{a-b}=\frac{(a-b)\,(a+b)}{a-b}=a+b$$

Check: $\frac{a+b-a}{a+b-b}=\frac{b}{a}$

$$\frac{b}{a}=\frac{b}{a}$$

Exercise 102

Solve for x and check:

1. $4a^3b^2x=15a^4b^3c$
2. $2mn^2x=2m^3n^2-6mn^3$
3. $(a+1)x=a^3+3a^2+3a+1$
4. $axy=b+c$
5. $ax+bx=a^4-b^4$
6. $2mx+3nx=5a$
7. $\frac{x+1}{x-1}=\frac{m+n}{m-n}$
8. $\frac{x}{m}+\frac{a}{n}=\frac{s}{t}$
9. $\frac{x}{a}+\frac{x}{b}-\frac{x}{c}=1$
10. $\frac{x}{a-c}-\frac{b}{a+c}=\frac{2bc}{a^2-c^2}$

153 In connection with formulas, it will be convenient to be able to solve a literal equation for *any* of the letters involved. For example: In the formula $C=2\pi r$, $r=\frac{C}{2\pi}$. In Exercise 85, such formulas were made into numerical equations by

substituting values for all letters but one, and then solving for that unknown. By treating them as literal equations, they can be solved for *any* letter, and the result can be used as a formula to obtain the value of that letter.

Example: Solve for x: $\frac{2a-x}{b}-\frac{a-x}{c}=\frac{a}{b}$

$$\frac{2a-x}{b}-\frac{a-x}{c}=\frac{a}{b}$$

$$2ac-cx-ab+bx=ac \qquad \text{Why?}$$

$$bx-cx=ab-ac \qquad \text{Why?}$$

$$x(b-c)=a(b-c) \qquad \text{Why?}$$

$$x=a \qquad (1)$$

Check: $\frac{2a-a}{b}-\frac{a-a}{c}=\frac{a}{b}$

$$\frac{a}{b}-0=\frac{a}{b}$$

$$\frac{a}{b}=\frac{a}{b}$$

Solve for b: $bx-ab=cx-ac$ Why?

$$b(x-a)=c(x-a) \qquad \text{Why?}$$

$$b=c \qquad \text{Why?} \qquad (2)$$

Check: $\frac{2a-x}{c}-\frac{a-x}{c}=\frac{a}{c}$

$$\frac{2a-x-a+x}{c}=\frac{a}{c}$$

$$\frac{a}{c}=\frac{a}{c}$$

$\left.\begin{array}{l}\text{Solve for a}\\ \text{Solve for c}\end{array}\right\}$ See (1) and (2)

Observe that the same steps are used as in Art. 151. Care must be taken in the last step to divide by the *coefficient of the required letter*.

Exercise 103

1. $x+a=2a-b$. Solve for x: for a: for b.
2. $5x-11m=3x+5m$. Solve for x: for m.
3. $x=ax+1$. Solve for x: for a.
4. $2x+a-b=2a+b-x$. Solve for x; for a; for b.
5. $\frac{a+3c}{4}-\frac{b}{2}=\frac{a}{2}+\frac{b}{3}-\frac{c}{4}$. Solve for a; for b; for c.
6. $\frac{m}{a}-\frac{m}{b}=3$. Solve for m; for a; for b.
7. $\frac{3a^2}{c}-\frac{x(a^2-c^2)}{c}=\frac{a(3a-1)}{c}+1$. Solve for x.
8. $\frac{a-3b}{a-2b}-\frac{a-2b}{a-3b}=\frac{5b(c+b)+cd}{a^2-5ab+6b^2}$. Solve for a; for b; for c; for d.

Monomial Factors

134 The process used in determining the polynomial coefficients of the unknown quantities in problems of Exercises 102 and 103 may be used in expressing a polynomial as the product of a monomial and a polynomial.

Example 1. $a^2-a=a(a-1)$

Example 2. $6a+8b=2(3a+4b)$

Example 3. $4x^2y+6xy^2-18xy=2xy(2x+3y-9)$

NOTE: The factors of Example 3 might be:

$$\left\{\begin{array}{l} 2 \text{ and } 2x^2y+3xy^2-9xy \\ x \text{ and } 4xy+6y^2-18y \\ y \text{ and } 4x^2+6xy-18x \\ 2x \text{ and } 2xy+3y^2-9y \\ 2y \text{ and } 2x^2+3xy-9x \\ xy \text{ and } 4x+6y-18 \\ 2xy \text{ and } 2x+3y-9 \end{array}\right.$$

but in such cases, it is better to take out the complete monomial factor.

Exercise 104

Factor:

1. x^3-x^2+x
2. $14m^4-21m^3+7m-7m^2$
3. $16x^2y^2z^2-4b^2c^2$
4. $a^2m^3-b^2m^3$
5. x^6-2x^3+1
6. $3m^4-6m^2+3$
7. $a^2-5ab-104b^2$
8. $c^5x^2+4c^5xy-21c^5y^2$
9. $40m^3+14m^2n-12mn^2$
10. $81a^4c^3d^5-625b^4c^3d^5$ (4 factors)

Reduce to lowest terms:

11. $\dfrac{3m^3n}{6m^2n^3}$
12. $\dfrac{2m-2n}{6a-6b}$
13. $\dfrac{2a+2b}{5a+5b}$
14. $\dfrac{ax-x}{bx+x}$
15. $\dfrac{12am+16bm}{18an+24bn}$
16. $\dfrac{a^2-3a+2}{2a^2x-6ax+4x}$
17. $\dfrac{16a^2m^3x^8-24a^2m^3x^7y}{24a^3mx^6+36a^3mx^5y}$
18. $\dfrac{12x^3-60x^2+72x}{3x^2y+9xy-30y}$

Exercise 105

Simplify:

1. $\dfrac{4m^3x^2}{7y^3}\cdot\dfrac{35y^5}{12m^2x^3}$
2. $\dfrac{3am-3m}{5am-5an}\cdot\dfrac{7bm-7bn}{4am-4bm}\cdot\dfrac{20a-20b}{21a+21}$
3. $\dfrac{2x+6}{3x-6}\cdot\dfrac{9x^2+27x-90}{20x^2+70x+30}$

4. $\dfrac{32abx^3-72ab^3x}{36mnx^2+6bmnx-90b^2mn} \div \dfrac{80bx^2+120b^2x}{27mnx^2+90bmnx+75b^2mn}$

5. $\dfrac{m}{2m-2n}+\dfrac{n}{5m-5n}$

6. $\dfrac{5}{4x-4y}-\dfrac{y}{x^2-y^2}$

7. $\dfrac{m}{12m^3n-8m^2n^2}-\dfrac{n}{15m^2n^2-10mn^3}$

8. $\dfrac{1}{4x^2-9y^2}-\dfrac{1}{4x^2+12xy+9y^2}$

9. $\dfrac{m+n}{16m^2-40mn+25n^2}+\dfrac{m-n}{16m^2-25n^2}$

10. $\dfrac{1}{x-y}+\dfrac{1}{x+y}-\dfrac{x}{x^2-y^2}+\dfrac{y}{x^2-2xy+y^2}$

Exercise 106

Solve for x:

1. $5ax-3a=5bx-3b$

2. $2a^2(3x-2)-2a(21x-10)=12(2-5x)$

3. $ab-\dfrac{abn}{m}=\dfrac{ab}{x}+\dfrac{abn}{mx}$

4. $\dfrac{2sx-t}{s}=\dfrac{7t^2+4stx-7s^2}{4t^2}$

5. $\dfrac{a}{a^2-ax}+\dfrac{3}{ab-bx}+\dfrac{3b+9}{2ab}=0$

6. $\dfrac{x}{3a^2-3ac}-\dfrac{1}{5ac+5c^2}=\dfrac{1}{a^2-c^2}$

7. $\dfrac{m^2+mn}{x+3n}+\dfrac{2nx(m+n)}{x^2+5nx+6n^2}=\dfrac{m^2+2mn+n^2}{x+2n}$

8. $\dfrac{1}{a^2-4b^2x^2}+\dfrac{1}{ab+2b^2x}=\dfrac{a^2+ab+b^2}{a^3b-4ab^3x^2}+\dfrac{1}{a^2-2abx}$

9. $$\frac{\frac{a}{2}+\frac{x}{3}}{\frac{a}{3}-\frac{x}{2}}-\frac{x}{2a+3x}=\frac{9x^2}{4a^2-9x^2}$$

10. $$\frac{b-1}{6x^2+bx-15b^2}+\frac{1}{8bx-12b^2}=\frac{1}{6x^2+10bx}+\frac{3x(1-2b)}{24bx^2+4b^2x-60b^3}$$

Exercise 107. Formulas as Literal Equations

1. $P=2(a+b)$. Solve for a. Use the result as a formula and evaluate when $P=17\frac{1}{8}$, $b=2\frac{1}{4}$.

2. $L=1\frac{3}{4}d+\frac{1}{8}$. Solve for d. Use the result as a formula and evaluate when $L=4\frac{1}{16}$.

3. $S=\frac{1}{2}gt^2+vt$. Solve for v. Evaluate the result when $S=164.72$, $t=3$.

4. $A=\frac{abc}{4r}$. Solve for r. Evaluate the result when $a=2.1$, $b=2.8$, $c=3.5$, $A=29.4$.

5. $V=\pi r^2a$. Solve for a. Evaluate the result when $V=46.9$, $r=2.3$.

6. $b^2=a^2+c^2+2a'c$. Solve for a'. Evaluate the result when $a=18$, $b=16$, $c=31$.

7. $b^2=a^2+c^2-2ac'$. Solve for c'. Evaluate the result when $a=14$, $b=15$, $c=16$.

8. $C=\frac{5}{9}(F-32)$. Solve for F. Evaluate the result when $C=25$.

9. $w=\frac{l}{h}\cdot p$. Solve for h. Evaluate the result when $w=150$, $l=162$, $p=100$.

10. $A=\frac{1}{2}h\,(b+b')$. Solve for h. Evaluate the result when $A=96$, $b=18$, $b'=6$.

11. $A + \frac{1}{2}r(a+b+c)$. Solve for a. Evaluate the result when $A=27.714$, $r=2.3095$, $p=8$, $c=8$.

12. $A=\frac{1}{2}(2\pi R+2\pi r)s$. Solve for r. Evaluate the result when $A=439.824$, $R=10$, $s=10$.

13. $V=\pi r^2 \cdot \frac{h+H}{2}$. Solve for H. Evaluate the result when $V=2246$, $r=8$, $h=6$.

14. $C=\frac{E}{R+r}$. Solve for R. Evaluate the result when $C=10$, $E=120$, $r=1.6$.

15. $F=\frac{uv}{u+v}$. Solve for v. Evaluate the result when $F=1\frac{7}{8}$, $u=3$.

Exercise 108

1. *a* times a number increased by *b* equals *c*. Find the number.

2. *m* times a number decreased by *n* equals *p*. Find the number.

3. There are three numbers whose sum is *s*. The second is *a* times the first, and the third is *b* times the first. Find the three numbers.

4. Divide *m* into two parts so that one is *r* more than the other.

5. Divide *m* into two parts such that one is *r* times the other.

6. The perimeter of a triangle is *p*. The first side is *a* less than the second, and the third is *b* times the first. Find the three sides.

7. The perimeter of a rectangle is *p*. The length is *a* more than *b* times the width. Find the dimensions.

8. The supplement of an angle is a times the complement. Find the angle.

9. The sum of the complement and the supplement of an angle is $b°$ less than a perigon. Find the angle.

10. The ratio of an angle to its complement is $\frac{a}{b}$. Find the angle.

11. The length of a rectangle is a more than its width, and the ratio of the length to the width is $\frac{2a}{b}$. Find the length.

12. The difference of the squares of two consecutive numbers is a. Find the numbers.

13. Divide a into two parts in the ratio $\frac{b}{c}$.

14. An alloy contains a parts of tin and b parts of copper. How many ounces of each in m oz. of the alloy?

15. The sides of a rectangle are in the ratio $\frac{l}{w}$, and the perimeter is p. Find the dimensions.

16. If a men can do a piece of work in n days, how long will it take b men to do it?

17. If a men can build n rods of fence in a given time, how many can b men build in the same time?

18. If the volume of a gas is v cubic centimeters under a pressure of a, find the pressure when the volume is v'.

19. The volume of a gas is v cubic centimeters when the absolute temperature is d. Find the temperature when the volume is v'.

20. A pulley whose diameter is a is belted to another whose diameter is b. If the R. P. M. of the second is c, find the R. P. M. of the first.

Quadratic Literal Equations

Example: Solve for x: $\frac{ax^2}{ab-b^2}+\frac{bx^2}{a^2-ab}=\frac{b^2c^5}{a}$

$$\frac{ax^2}{b(a-b)}+\frac{bx^2}{a(a-b)}=\frac{b^2c^5}{a} \qquad \textbf{Why?}$$

$$a^2x^2+b^2x^2=b^3c^5(a-b) \qquad \textbf{Why?}$$

$$x^2(a^2+b^2)=b^3c^5(-b) \qquad \textbf{Why?}$$

$$x^2=\frac{b^3c^5\,(a-b)}{a^2+b^2} \qquad \textbf{Why?}$$

$$x=\pm\sqrt{\frac{b^3c^5(a-b)}{a^2+b^2}} \qquad \textbf{Why?}$$

NOTE: The result is left as the indicated square root because the value of x^2 is not a perfect square.

Exercise 109

Solve for x:

1. $3m^2x^2+3m^4=7m^4-m^2x^2$
2. $ax^2-b=c$
3. $(mx-n)^2+(mx+n)^2=10n^2$
4. $(x-r)(x+s)+(x-s)(x+r)=2(r^2+rs+s^2)$
5. $\frac{x^2}{x^2-2c}=\frac{8}{3}$
6. $1+\frac{5x+2a}{3x}=\frac{2a}{a-x}$
7. $\frac{x^2-a}{x^2+a}=\frac{c-d}{c+d}$
8. $\frac{m}{n-x^2}=\frac{n}{m-x^2}$
9. $\frac{x+2y}{x-2y}+\frac{x-2y}{x+2y}=5$
10. $\frac{4x^2}{x^2+3b}-\frac{6b}{2x^2+6b}=\frac{10x^2+1}{3x^2+9b}$.

Exercise 110. Formulas as Quadratic Literal Equations

1. $A=\pi r^2$. Solve for r.
2. $V=\pi r^2 a$. Solve for r.
3. $c^2=a^2+b^2$. Solve for a.
4. $S=\frac{1}{2}gt^2$. Solve for t.
5. $A=\frac{\pi D^2}{4}$. Solve for D.
6. $b^2=a^2+c^2+2a'c$. Solve for a.
7. $a^2+b^2=2\left(\frac{c}{2}\right)^2+m^2$. Solve for m.
8. $E=\frac{MV^2}{2a}$. Solve for V.
9. $V=\pi r^2\frac{h+H}{2}$. Solve for r.
10. $a^2+b^2=2\left(\frac{c}{2}\right)^2+m^2$. Solve for c.

Exercise 111

1. Find the hypotenuse of a right triangle whose sides are $3a$ and $4a$.

2. Find one side of a right triangle whose hypotenuse and other side are respectively $3c$ and $5b$.

3. Find the diagonal of a square whose side is s.

4. Find the side of a square whose diagonal is d.

5. Find the sides of a rectangle whose diagonal is d if they are in the ratio $\frac{a}{b}$.

Quadratic Literal Equations Solved by Factoring

155 Many quadratic literal equations may be solved by the method explained in Art. 141.

Example: Solve for x: $\frac{x}{3b}-\frac{7}{12}=\frac{b}{4x}-\frac{x}{6b}$

$$\frac{x}{3b}-\frac{7}{12}=\frac{b}{4x}-\frac{x}{6b}$$

$$4x^2-7bx=3b^2-2x^2 \qquad \text{Why?}$$

$$6x^2-7bx-3b^2=0 \qquad \text{Why?}$$

$$(3x+b)(2x-3b)=0 \qquad \text{Why?}$$

$$3x+b=0 \qquad 2x-3b=0$$

$$x=\frac{-b}{3} \quad \text{or} \quad x=\frac{3b}{2}$$

Exercise 112

Solve for x:

1. $\frac{y^2}{4}=\frac{x^2}{9}$
2. $\frac{x^2}{2a}-x=\frac{3a}{2}$
3. $\frac{x^2}{6}+8r^2=\frac{8rx}{3}$
4. $\frac{6(x-m)}{5}=\frac{mx}{x+m}$
5. $\frac{x}{x-3b}+\frac{b}{x-2b}=\frac{27b^2}{x^2-5bx+6b^2}$
6. $\frac{x}{a(a-x)}=\frac{6}{7(x+5a)}$
7. $\frac{9(x-3a)}{4(x-a)}=\frac{x-a}{x-3a}$
8. $\frac{x-m}{x^2-4mx+4m^2}+\frac{1}{x-m}=\frac{2x-5m}{2x^2-6mx+4m^2}$

Quadratic Literal Equations Solved by Completing the Square

156 Quadratic literal equations which are not readily put into their factored form, may be solved by completing the square using the same method as used in numerical equations explained in Art. 133.

Example: Solve for x: $\frac{5x}{a^2+3ab+2b^2}=\frac{20}{x+2b}$

$$\frac{5x}{a^2+3ab+2b^2}=\frac{20}{x+2b}$$

$5x^2+10bx=20a^2+60ab+40b^2$ Why?

$x^2+2bx=4a^2+12ab+8b^2$ Why?

$x^2+2bx+b^2=4a^2+12ab+9b^2$ Why?

$x+b=\pm(2a+3b)$ Why?

$x=2a+2b$ or $-2a-4b$ Why?

Exercise 113

Solve for x:

1. $ax^2+2bx=\frac{8b^2}{a}$
2. $\frac{m^3x-2am^3}{2a+b}=\frac{bm^3}{x}$
3. $\frac{4x}{b-5a}=\frac{a+b}{x-3a}$
4. $x^2-3ax-1+3a=0$
5. $\frac{x^2-4a^2}{b}=x+2a$
6. $x^2-\frac{2m+1}{n^2}=\frac{2mx}{n}$
7. $x^2-\frac{2x}{ab}=4\cdot\frac{ab-1}{ab}$
8. $x^2=\frac{9a^2+3ab}{c^2}+\frac{b}{c}\cdot x$
9. $4m^2x^2-n^2=4m^2x-m^2$
10. $(m^2+n^2)x^2+2(m^2+n^2)x+\frac{(m^2-n^2)^2}{m^2+n^2}=0$

157 In numerical quadratic equations, it is always possible to express the results without the radical sign (See Art. 133, Example 2). In literal quadratic equations, it is sometimes necessary to leave the results in a form involving radicals.

For example:

$$x^2+2ax+a^2=b^3$$
$$x+a=\pm\sqrt{b^3}$$
$$x=\pm\sqrt{b^3}-a$$

These radicals should be expressed in their *simplest form.*

158 A radical is said to be in its simplest form when:

1. The expression under the radical sign contains no numerical factor.

2. The expression under the radical sign contains no factor which is a perfect square.

3. The expression under the radical sign is not a fraction.

Example 1: Simplify $\sqrt{5a}$

$$\sqrt{5a}=\pm 2.2361\sqrt{a} \text{ (See Art. 121)}$$

Example 2: Simplify $\sqrt{12a^5b^3m^2+48a^5b^3mn+48a^5b^3n^2}$

$$\sqrt{12a^5b^3m^2+48a^5b^3mn+48a^5b^3n^2}=$$
$$\sqrt{12a^5b^3(m^2+4mn+4n^2)}=$$
$$\sqrt{4\cdot 3a^4b^2(m+2n)^2}\cdot\sqrt{ab}=$$
$$\pm 2\cdot 1.7321a^2b(m+2n)\sqrt{ab}=$$
$$\pm 3.464a^2b(m+2n)\sqrt{ab} \text{ (See Art. 121)}$$

Example 3: Simplify $\sqrt{\dfrac{3a^5b^3}{5m^2n^5}}$

$$\sqrt{\frac{3a^5b^3}{5m^2n^5}}=\sqrt{\frac{3a^5b^3}{5m^2n^5}\cdot\frac{5n}{5n}} \quad \text{(See Art. 123)}$$
$$=\sqrt{\frac{15a^5b^3n}{25m^2n^6}}$$
$$=\sqrt{\frac{15a^4b^2}{25m^2n^6}}\cdot\sqrt{abn} \quad \text{(See Art. 121)}$$
$$=\pm\frac{3.87a^2b}{5mn^3}\cdot\sqrt{abn}$$

Exercise 114

Simplify:

1. $\sqrt{48a^5b^2}$
2. $\sqrt{192a^3x^5}$
3. $\sqrt{12(a-x)y^3}$
4. $\sqrt{242(x-y)^2(a+b)}$
5. $\sqrt{(a-b)^3(x+m)^2(y-z)^5}$
6. $\sqrt{3x^2-6xy+3y^2}$
7. $\sqrt{\frac{3}{5}a^7b^2}$
8. $\sqrt{\dfrac{a-x}{3}}$
9. $\sqrt{\frac{1}{7}x^3(a+b)^3}$
10. $\sqrt{\dfrac{a^4x-a^4y}{5}}$
11. $\sqrt{\dfrac{9m^3x^2-30m^3xz+25m^3z^2}{4}}$
12. $\sqrt{\dfrac{3m^2n^5}{x}}$
13. $\sqrt{\dfrac{4a^3b}{5c}}$
14. $\sqrt{\dfrac{5a}{x-y}}$
15. $\sqrt{\dfrac{m^3y^2}{a^2(a-b)}}$
16. $\sqrt{\dfrac{a-b}{a+b}}$
17. $\sqrt{1-\dfrac{a^2}{c^2}}$
18. $\sqrt{\dfrac{a^2}{b^2}+1}$
19. $\sqrt{1+\dfrac{b}{a-b}}$
20. $\sqrt{\dfrac{a}{b}-\dfrac{c}{d}}$
21. $\sqrt{\dfrac{a}{b}+\dfrac{b}{a-2b}}$
22. $\sqrt{\dfrac{x}{x-y}-\dfrac{y}{x+y}}$
23. $\sqrt{1+\dfrac{4ab}{(a-b)^2}}$
24. $\sqrt{\dfrac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{x}+\frac{1}{y}}}$
25. $\sqrt{\dfrac{\frac{a}{b}-\frac{x}{y}}{1-\frac{ax}{by}}}$

Example: Solve for x: $ax^2-3x=9$

$$ax^2-3x=9$$

$$x^2-\frac{3x}{a}=\frac{9}{a} \qquad \text{Why?}$$

$$x^2-\frac{3x}{a}+\frac{9}{4a^2}=\frac{9}{a}+\frac{9}{4a^2}=\frac{36a+9}{4a^2} \qquad \text{Why?}$$

$$x-\frac{3}{2a}=\pm\sqrt{\frac{36a+9}{4a^2}}$$

$$x-\frac{3}{2a}=\pm\frac{3}{2a}\sqrt{4a+1}$$

$$x=\frac{3}{2a}\pm\frac{3}{2a}\sqrt{4a+1}$$

$$x=\frac{3\pm3\sqrt{4a+1}}{2a}$$

Exercise 115

Solve for x:

1. $4x^2-4ax=b^2m-a^2$
2. $9bx(bx-2)=a^3-9$
3. $\frac{x-3m}{m-8}=\frac{3m^2}{4x}$
4. $\frac{x^2}{a}-\frac{1}{ab}=2x-a$
5. $(3x+2m)^2=\frac{a^2}{b^3}$
6. $\frac{a(2x^2+1)}{12b}+\frac{1}{4}=\frac{b+ax}{3b}$
7. $\frac{c^2(x-1)}{6x-9b}=\frac{c^2b}{x+1}-\frac{a^2}{(x+1)(6x-9b)}$
8. $\frac{a-bx^2}{8dmx+16dm^2+c}=\frac{b}{d}$

9. $\dfrac{x+10ny}{m-n}=\dfrac{m-n}{x}-\dfrac{25n^2y^2}{mx-nx}$

10. $\dfrac{1}{(a-b)^2}-\dfrac{6by}{a^3x-2a^2bx+ab^2x}=\dfrac{27m^3}{a^2x^2}-\dfrac{9b^2y^2}{a^2x^2(a-b)^2}$

The Quadratic Formula

159 All quadratic equations can be arranged in the form $ax^2+bx=c$. The solution of this equation is as follows:

$$ax^2+bx=c$$

$$x^2+\frac{bx}{a}=\frac{c}{a} \qquad \text{Why?}$$

$$x^2+\frac{bx}{a}+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}+\frac{c}{a} \qquad \text{Why?}$$

$$x^2+\frac{bx}{a}+\frac{b^2}{4a^2}=\frac{b^2+4ac}{4a^2} \qquad \text{Why?}$$

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2+4ac}}{2a}$$

$$x=\frac{-b\pm\sqrt{b^2+4ac}}{2a}$$

Since this equation represents *every* quadratic equation, the result may be used as a formula for solving any quadratic equation.

Example 1: Solve by the formula:

$$3+\frac{15}{2x^2+x-6}=\frac{3}{4x-6}+\frac{1}{x+2}$$

$$3+\frac{15}{(2x-3)(x+2)}=\frac{3}{2(2x-3)}+\frac{1}{x+2} \qquad \text{Why?}$$

$$3\cdot 2(2x-3)(x-2)+15\cdot 2=3(x+2)+2(2x-3) \qquad \text{Why?}$$

$$12x^2+6x-36+30=3x+6+4x-6 \qquad \text{Why?}$$

$$12x^2-x=6 \qquad \text{Why?}$$

Evaluating the formula $x=\frac{-b\pm\sqrt{b^2+4ac}}{2a}$ for $a=12$, $b=-1$, $c=6$.

$$x=\frac{+1\pm\sqrt{1+4\cdot12\cdot6}}{2\cdot12}$$

$$x=\frac{1\pm\sqrt{289}}{24}$$

$$x=\frac{1\pm17}{24}=\frac{18}{24}\text{ or }\frac{-16}{24}$$

$$x=\frac{3}{4}\text{ or }\frac{-2}{3}$$

Example 2: Solve by the formula:

$$\frac{x}{3}-\frac{a}{6m}+\frac{a^2}{2m^3x}=0$$

$$2m^3x^2-am^2x+3a^2=0 \qquad \text{Why?}$$

$$2m^3x^2-am^2x=-3a^2 \qquad \text{Why?}$$

Evaluating the formula $x=\frac{-b\pm\sqrt{b^2+4ac}}{2a}$

for $a=2m^3$; $b=-am^2$; $c=-3a^2$.

$$x=\frac{-(-am^2)\pm\sqrt{(-am^2)^2+4\cdot(2m^3)\cdot(-3a^2)}}{2\cdot2m^3}$$

$$x=\frac{am^2\pm\sqrt{a^2m^4-24a^2m^3}}{4m^3}$$

$$x=\frac{am^2\pm am\sqrt{m^2-24m}}{4m^3}$$

$$x=\frac{am\pm a\sqrt{m^2-24m}}{4m^2}$$

Example 3: Solve by the formula:

$$\frac{x+5}{x-5}-\frac{x-5}{x+5}=\frac{3x^2}{x^2-25}$$

$$(x+5)^2-(x-5)^2=3x^2 \qquad \text{Why?}$$

$$x^2+10x+25-x^2+10x-25=3x^2 \qquad \text{Why?}$$

$$20x=3x^2 \qquad \text{Why?}$$

$$3x^2-20x=0 \qquad \text{Why?}$$

$$x=\frac{-(-20)\pm\sqrt{(-20)^2-4\cdot3\cdot0}}{2\cdot3}$$

$$x=\frac{20\pm\sqrt{400+0}}{6}$$

$$x=\frac{20\pm20}{6}=\frac{40}{6}\text{ or }\frac{0}{6}$$

$$x=6\tfrac{2}{3}\text{ or }0$$

NOTE: If a quadratic equation after being simplified lacks a term, then the coefficient of that term is zero.

Exercise 116

Solve by the formula:

1. $(x-2)^3-(x-7)^3=50$
2. $\dfrac{x+2}{x-1}-\dfrac{7}{3}=\dfrac{4-x}{2x}$
3. $\dfrac{x-3b}{x-a}=\dfrac{4a}{x+3b}$
4. $\dfrac{24}{1-x}=12+\dfrac{1}{1-x^2}$
5. $\dfrac{x-7a}{3a+4b}=\dfrac{a}{x-7a}$
6. $\dfrac{cx}{3a}+\dfrac{a^3}{12cx}=\dfrac{9b^2m}{12acx}+\dfrac{a}{3}$
7. $\dfrac{x}{x-1}+\dfrac{4}{5(20x^2-37x+17)}=\dfrac{3}{20x-17}$
8. $\dfrac{x-1}{6x^2+6x-36}=\dfrac{x+1}{3x+9}-\dfrac{1}{2x-4}$
9. $\dfrac{x^2+3x+9}{72x^2+66x-30}-\dfrac{x-3}{12x+15}=\dfrac{x-3}{12x-4}$
10. $(5x+4)(x-2)+(11x+7)(x-3)=-30$

Exercise 117

1. The length of a rectangle is a more than its width and the area is b. Find the dimensions.

2. The product of two consecutive numbers is m. Find them.

3. The sum of the squares of two consecutive numbers is s. Find them.

4. The difference of the squares of two consecutive numbers is d. Find them.

5. The hypotenuse of a right triangle is c and one side is m more than the other. Find the sides.

6. The diagonal of a rectangle is d and its length is a more than its width. Fina the dimensions.

7. The sum of the squares of three consecutive numbers is n Find the numbers.

8. The sum of the areas of two squares is s and the side of one is n more than a side of the other. Find the sides of both squares.

9. The length of a rectangle is a more than b times its width. The area is A. Find the dimensions.

10. Divide n into two parts whose product shall be p.

11. The square of the larger of two consecutive numbers is d more than twice the smaller number. Find the numbers.

12. The side of one square is $\frac{a}{b}$ times that of another, and the sum of their perimeters is 96. Find the sum of their areas.

13. The perimeter of a rectangle is p and its area is A. Find the dimensions.

14. The area of a circle is 3.1416A. Find the radius.

15. The sum of the areas of two circles is b and the radius of one is a more than the radius of the other. Find the radii.

Use π as a general number throughout the problem.

CHAPTER XII

MISCELLANEOUS EQUATIONS

Equations Solved like Quadratics

160 Any equation that can be reduced to the form $ax^{2m}+bx^m=c$ can be solved by the methods used in solving quadratics; for $ax^{2m}+bx^m=c$ may be expressed $a(x^m)^2+b(x^m)=c$ and may be solved as a quadratic in which the unknown is x^m.

Example 1: Solve: $2x^4 \quad +x^2=6$

$$2(x^2)^2+x^2=6$$

Evaluating the formula $x=\dfrac{-b\pm\sqrt{b^2+4ac}}{2a}$ for $x=x^2$, $a=2$, $b=1$, $c=6$

$$x^2=\frac{-1\pm\sqrt{1+4\cdot2\cdot6}}{2\cdot2}$$

$$x^2=\frac{-1\pm7}{4}=\frac{3}{2}\text{ or }-2$$

$$x=\pm\sqrt{\frac{3}{2}}\text{ or }\pm\sqrt{-2}$$

$$x=\pm1.225\text{ or }\pm\sqrt{-2}$$

Observe:

I. By solving the equation as a quadratic, two quadratic equations are obtained $x^2=\frac{3}{2}$, $x^2=-2$ and the solution of these gives

$$x=+1.225,\ -1.225,\ +\sqrt{-2},\ -\sqrt{-2}$$

II. A *fourth* power equation has *four* values of the unknown.

III. The square root of a negative number can not be found (see note Art. 113) and therefore two of these results ($\pm\sqrt{-2}$) are of no practical use. Such numbers are called *imaginary numbers*.

Example 2: Solve: $x^6-7x^3-8=0$

$$(x^3)^2-7x^3=8$$

Evaluating the formula $x=\dfrac{-b\pm\sqrt{b^2+4ac}}{2a}$

for $x=x^3$, $a=1$, $b=-7$, $c=8$

$$x^3=\frac{-(-7)\pm\sqrt{(-7)^2+4\cdot1\cdot8}}{2\cdot1}$$

$$x^3=\frac{7\pm9}{2}=8 \text{ or } -1$$

$$x=\sqrt[3]{8} \text{ or } \sqrt[3]{-1}$$

$$x=2 \text{ or } -1$$

Observe:

I. By solving as a quadratic equation, two equations are obtained, $x^3=8$ and $x^3=-1$.

II. These two equations are solved by extracting the *cube root* of both members.

III. $\sqrt[3]{8}=+2$ because $(+2)^3=(+2)(+2)(+2)=+8$

$\sqrt[3]{-1}=-1$ because $(-1)^3=(-1)(-1)(-1)=-1$

NOTE: A **sixth** power equation will always have **six** values of the unknown but in problems of the form of example 2, the other four contain imaginary numbers and hence may be disregarded in practical problems.

Exercise 118

Solve for x:

1. $x^4+108=21x^2$
2. $12(x^4-x^2+1)-5(x^2+2)=0$
3. $x^3(\frac{x^3}{2}+2)+2(x^3-\frac{1}{4})=\frac{x^3}{16}$
4. $\dfrac{x^3+4}{3}=\dfrac{31}{30-x^3}$

5. $\dfrac{x^2}{x+3}+\dfrac{3(x-3)}{4x^2}=0$

6. $\dfrac{2}{3x^2+2}+\dfrac{5}{x^2(x^2-3)}=0$

7. $\dfrac{x^4}{a^4b^4}-\dfrac{4(x^2-a^2b^2)}{a^2b^2}=\dfrac{x^2}{a^2b^2}+2$

8. $\dfrac{x^2-2m^2}{m^2x^2}=\dfrac{7mx}{x^4+2m^2x^2+4m^4}$

9. $\dfrac{3}{8x^6+24a^3x^3+10a^6}=\dfrac{2}{8x^6-5a^3x^3+4a^6}$

10. $\dfrac{10(3x^2+m^2)}{m^2x^2}+\dfrac{2}{m^2}=\dfrac{25}{x^2}-\dfrac{1}{3x^2-m^2}$.

Equations Containing Radicals

161 A *radical equation* is one which contains one or more indicated roots of expressions involving the unknown.

Example 1: Solve: $\sqrt{x}=3$

$x=9$ (squaring both members)

Note. $(\sqrt{x})^2=x$. (See Art. 113.)

Example 2: Solve: $\sqrt{x-7}-x+9=0$

$$\sqrt{x-7}-x+9=0$$

$$\sqrt{x-7}=x-9 \qquad \text{Why?}$$

$$x-7=x^2-18x+81 \text{ (Squaring both members)}$$

$$x^2-19x=-88 \qquad \text{Why?}$$

$$x=\frac{19\pm\sqrt{361-352}}{2}$$

$$x=\frac{19\pm3}{2}=11 \text{ or } 8$$

Check: $\sqrt{11-7}-11+9=0$ $\quad$ $\sqrt{8-7}-8+9=0$

$2-11+9=0$ or $\quad$ $1-8+9=0$

$0=0$ $\quad$ $2=0$

Does not check

Observe.

I. An equation of the form $\sqrt{x}=a$ may be cleared of its radical sign by squaring both members.

II. The equation must be arranged so that the radical stands *alone* in one member *before squaring.*

III. All values of the unknown may *not* check in the original equation. This is due to the fact that *when both members of an equation are squared, new solutions may be introduced and it is necessary therefore to retain only those values which check in the original equation.*

Exercise 119

Solve and check:

1. $\sqrt{x}=2$
2. $\sqrt{x}=x-2$
3. $\sqrt{x+1}=3$
4. $\sqrt{x+3}=\frac{x-3}{5}$
5. $8-\sqrt{x-1}=6$
6. $x+\sqrt{x^2+3}=1$
7. $\sqrt{x^2+4x+9}=x+7$
8. $\sqrt{5x^2-4}=2x+3$
9. $\frac{\sqrt{x+2}}{6}=\frac{x-1}{3}$
10. $\frac{\sqrt{x^2+9}}{5}=\frac{\sqrt{x}}{2}$

162 If equations contain more than one radical, a knowledge of addition, subtraction and multiplication of radicals may be necessary for their solution.

Addition and Subtraction of Radicals

163 Similar Radicals are those which differ only in their coefficients.

164 Only similar radicals can be combined.

Example: $2\sqrt{3}-5\sqrt{3}+7\sqrt{3}=4\sqrt{3}$

$a\sqrt{b}+c\sqrt{b}-d\sqrt{b}=(a+c-d)\sqrt{b}$

$\sqrt{12}+\sqrt{18}+2\sqrt{27}-\sqrt{75}=$

$2\sqrt{3}+3\sqrt{2}+6\sqrt{3}-5\sqrt{3}=3\sqrt{3}+3\sqrt{2}$

Exercise 120

Combine:

1. $\sqrt{50}-\sqrt{32}$
2. $\sqrt{20}-\sqrt{5}+\sqrt{45}$
3. $5\sqrt{3}-2\sqrt{27}$
4. $2\sqrt{\frac{3}{4}}+\sqrt{48}-\sqrt{72}$
5. $3\sqrt{18}-2\sqrt{50}+2\sqrt{18}-5\sqrt{2}$
6. $\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{6}}$
7. $\sqrt{\frac{3}{4}}+\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}$
8. $\sqrt{2a}+\sqrt{8a}-\frac{1}{3}\sqrt{27a}$
9. $2\sqrt{a^3m^2}-\sqrt{a^5}+3\sqrt{am^4}$
10. $\sqrt{a^2b^2+b^4}+\sqrt{a^4+a^2b^2}$
11. $\sqrt{2x+1}-\sqrt{2x}+\sqrt{8x+4}-\sqrt{8x}$
12. $\sqrt{144}+\sqrt{8x}-\sqrt{18x}+\sqrt{x^2}$

Multiplication of Radicals

165 By Art. 121, the product of two square roots is equal to the square root of their product.

Example 1: $2\sqrt{2}\cdot 4\sqrt{3}=8\sqrt{6}$

Example 2: $\sqrt{a}\cdot\sqrt{x}\cdot\sqrt{x+1}=\sqrt{ax^2+ax}$

Example 3: $3\sqrt{x+1}\cdot 2\sqrt{x-2}=6\sqrt{x^2-x-2}$

Example 4: $(2\sqrt{2}+3\sqrt{3x})(5\sqrt{2}-4\sqrt{3x})=$

$$\begin{array}{llll}
2\sqrt{2} & +3\sqrt{3x} & & \\
5\sqrt{2} & -4\sqrt{3x} & & \\
\hline
20 & +15\sqrt{6x} & & \\
 & -\ 8\sqrt{6x} & -36x & \\
\hline
20 & +\ 7\sqrt{6x} & -36x &
\end{array}$$

Example 5: $(2\sqrt{x}-3\sqrt{x-1})^2=$

$(2\sqrt{x})^2-2(2\sqrt{x})(3\sqrt{x-1})+(3\sqrt{x-1})^2=$

$4x-12\sqrt{x^2-x}+9x-9=13x-12\sqrt{x^2-x}-9$

Exercise 121

Multiply:

1. $2\sqrt{15}\cdot 3\sqrt{5}$
2. $8\sqrt{12a}\cdot 3\sqrt{24ab}$
3. $(\sqrt{5}+\sqrt{10})\sqrt{5}$
4. $(\sqrt{x}+\sqrt{x+2})\sqrt{x}$
5. $(\sqrt{x+3}+\sqrt{3x-4})\sqrt{3x-4}$
6. $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$
7. $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$
8. $(\sqrt{12}+2\sqrt{2})(\sqrt{27}-\sqrt{18})$
9. $(a+\sqrt{b})(a-3\sqrt{b})$
10. $(\sqrt{3}+\sqrt{5}+4\sqrt{2})(2\sqrt{3}-3\sqrt{5})$
11. $(\sqrt{3}-\sqrt{5})^2$
12. $(4-3\sqrt{2})^2$
13. $(5\sqrt{5x}-3\sqrt{3y})^2$
14. $(\sqrt{x}-\sqrt{x+y})^2$
15. $(3\sqrt{m+n}-2\sqrt{m-2n})^2$

Radical Equations Continued

Example: Solve:

$$2+\sqrt{2x+7}-\sqrt{5x+4}=0$$

$$2+\sqrt{2x+7}=\sqrt{5x+4} \qquad \textbf{Why?}$$

$$4+4\sqrt{2x+7}+2x+7=5x+4 \qquad \text{(squaring both members)}$$

$$4\sqrt{2x+7}=3x-7 \qquad \textbf{Why?}$$

$$32x+112=9x^2-42x+49 \qquad \text{(squaring both members)}$$

$$9x^2-74x-63=0 \qquad \textbf{Why?}$$

$$x=9 \text{ or } -\tfrac{7}{9}$$

Check:

$$
\begin{array}{llll}
2+\sqrt{18+7}-\sqrt{45+4}=0 & & 2+\sqrt{-\frac{14}{9}+7}-\sqrt{-\frac{35}{9}+4}=0 \\
2+\quad 5 \quad - \quad 7 \quad =0 & \text{or} & 2+\quad \frac{7}{3} \quad - \quad \frac{1}{3}=0 \\
\qquad\qquad 0=0 & & \qquad\qquad 4=0
\end{array}
$$

Does not check

$$\therefore x=9 \text{ but } x \neq -\tfrac{7}{9}$$

NOTE: **The sign $\neq$ means does not equal.**

Observe that if an equation contains more than one radical, it should be arranged so that the most complex radical stands *alone* in one member *before squaring.*

Exercise 122

Solve for x and check:

1. $\sqrt{x+4}=3-\sqrt{x}$
2. $\sqrt{2x-7}+\sqrt{2x+9}=8$
3. $\sqrt{x}-\sqrt{x-3}=\dfrac{2}{\sqrt{x}}$.
4. $\sqrt{8x+17}-\sqrt{2x}=\sqrt{2x+9}$
5. $\sqrt{3x+7}-\sqrt{x+1}-2\sqrt{x-2}=0$
6. $2\sqrt{5x}-\sqrt{2x-1}=\dfrac{4x+1}{\sqrt{2x-1}}$
7. $\dfrac{12}{\sqrt{x^2+8}}=\sqrt{x^2+8}+x$
8. $\sqrt{x-m^2}-\sqrt{x+7m^2}+\sqrt{x+2m^2}=0$
9. $3\sqrt{x^3+17}-2\sqrt{5x^3+41}+\sqrt{x^3+1}=0$
10. $\dfrac{\sqrt{x}-1}{\sqrt{x}+1}=\dfrac{x}{x-1}$

Factor Theorem

166 Equations of a higher power than a quadratic can not always be put in the form $ax^{2m}+bx\ =c$. In that case they can *not* be solved by the quadratic formula. By Article 136, if such an equation can be expressed in its factored form, it can be solved by placing each factor equal to zero and by solving each equation thus formed separately. For example:

$x^3-x^2-14x+24=0$ can be solved if it can be discovered that

$$x^3-x^2-14x+24=(x-2)(x-3)(x+4)$$

Then $(x-2)(x-3)(x+4)=0$

$$x=2,\ 3,\quad -4$$

167 The solution of such equations then requires the ability to factor such expressions. This is done by the principle called the *Factor Theorem.*

Let it be required to solve $x^4-x^3-7x^2+x+6=0$

Evaluating the equation for $x=+1$, $1-1-7+1+6=0$
$0=0$

$x=+1$ checks the equation. Therefore $x-1$ must be one factor.

Evaluating the equation for $x=-1$, $1+1-7-1+6=0$
$0=0$

$x=-1$ checks the equation. Therefore $x+1$ must be one factor.

Evaluating the equation for $x=+2$, $16-8-28+2+6=0$
$-12\neq 0$

$x=+2$ does *not* check the equation. Therefore $x-2$ is *not* a factor.

Evaluating the equation for $x=-2$, $16+8-28-2+6=0$
$0=0$

$x=-2$ checks the equation. Therefore $x+2$ is one factor.

Evaluating the equation for $x=+3$, $81-27-63+3+6=0$

$$0=0$$

$x=+3$ checks the equation. Therefore $x-3$ is one factor.

The four factors of $x^4-x^3-7x^2+x+6$ are, therefore, $(x-1)(x+1)(x+2)(x-3)$, and $x=+1, -1, -2, +3$.

Observe that the numbers used for evaluating the equation are factors of 6.

168 Factor Theorem: If an expression involving x becomes zero when evaluated for $x=a$, then $x-a$ is a factor of the given expression.

Example 1: Solve: $x^3+2x^2-12x-9=0$

The only numbers which are likely to reduce the expression to zero are ± 1, ± 3, ± 9. Of these $+3$ is the only one which checks the given equation. Dividing the given expression by $x-3$ shows that $(x-3)(x^2+5x+3)$ are the factors of $x^3+2x^2-12x-9$.

$$\text{Therefore } (x-3)(x^2+5x+3)=0$$

$$x-3=0 \quad \text{or} \quad x^2+5x+3=0$$

$$x=3 \qquad x^2+5x=-3$$

$$x=\frac{-5\pm\sqrt{25-12}}{2}$$

$$x=\frac{-5\pm\sqrt{13}}{2}$$

Observe:

I. If all the factors can not be discovered by the factor theorem, the remaining factor can be found by dividing by the known factors.

II. If the remaining factor produces a first or second power equation, it can be solved by methods already shown.

Example 2: Solve: $4x^4+4x^3-17x^2-9x+18=0$

The only numbers which are likely to reduce the expression to zero are ±1, ±2, ±3, ±6, ±9, ±18. Of these $+1$, and -2 are the only ones which check the equation.

$$\text{Therefore } (x-1)(x+2)(4x^2-9)=0$$

$$x-1=0 \quad \text{or} \quad x+2=0 \quad \text{or} \quad 4x^2-9=0$$

$$x=1 \qquad x=-2 \qquad 4x^2=9$$

$$x^2=\tfrac{9}{4}$$

$$x=\pm\tfrac{3}{2}$$

$$\therefore x=1,\ -2,\ \pm\tfrac{3}{2}$$

Exercise 123

Solve:

1. $x^3+2x^2-5x-6=0$
2. $x^3-2x^2-5x+6=0$
3. $x^3-7x+6=0$
4. $x^3-7x-6=0$
5. $x^4-2x^2+1=0$
6. $9x^4+4=13x^2$
7. $2x^3-3x^2-39x+20=0$
8. $x-1=\dfrac{1}{x^2+x+1}$
9. $\dfrac{x^2}{4-x}=\dfrac{4+x}{x^2-1}$
10. $x^2-x+4+\sqrt{x^2-x+4}=2$

11. $x^2-7x+\sqrt{x^2-7x+18}=24$

12. $x^3-3ax^2+3a^2x-a^3=0$

13. $x^3-ax^2+bx^2+x^2-abx-ax+bx-ab=0$

14. $3\left(\frac{x}{6}+\frac{2}{x}\right)^2-2\left(\frac{x}{6}+\frac{2}{x}\right)=\frac{8}{3}$

15. $\sqrt{3x^2-16x+21}=\dfrac{28}{\sqrt{3x^2-16x+21}}-3$

Quadratic Equations Continued

169 Literal quadratic equations may have more than one term containing respectively the first and second powers of the unknown. In such cases, it is necessary to express the equation with polynomial coefficients before evaluating the formula.

Example: Solve for x: $2x^2+bx+a^2=ab+3ax$

$2x^2+bx-3ax=ab-a^2$

$2x^2+x(b-3a)=ab-a^2$

Evaluating the formula for $a=2$, $b=b-3a$, $c=ab-a^2$

$$x=\frac{-(b-3a)\pm\sqrt{(b-3a)^2+4\cdot 2\cdot(ab-a^2)}}{2\cdot 2}$$

$$x=\frac{-b+3a\pm\sqrt{b^2-6ab+9a^2+8ab-8a^2}}{4}$$

$$x=\frac{-b+3a\pm\sqrt{b^2+2ab+a^2}}{4}$$

$$x=\frac{-b+3a\pm(b+a)}{4}$$

$$x=\frac{-b+3a+b+a}{4}\text{ or }\frac{-b+3a-b-a}{4}$$

$$x=a\text{ or }\frac{a-b}{2}$$

Exercise 124

Solve for x:

1. $(a-2)x^2-ax=-2$

2. $2x^2+(3a-8b)x=12ab$

3. $x^2-ax-bx+ab=0$

4. $a-x^2=(1-a)x$

5. $b(x-b)+a(x-a)=2(x-a)(x-b)$

6. $(1-c^2)(x+c)=2c(1-x^2)$

7. $\sqrt{8mn+x(m-2n)}-4n=x$

8. $\dfrac{(x^2+1)(a-b)}{x}=\dfrac{2a^2+2b^2}{a+b}$

9. $\dfrac{1}{x-m}-\dfrac{m+n}{mn}=\dfrac{1}{n-x}$

10. $\dfrac{x^2}{a+b}+\dfrac{2b}{a^2-b^2}=\dfrac{x}{a-b}$

Exercise 125. Formulas as Quadratic Literal Equations

1. $S=\frac{1}{2}gt^2+vt$. Solve for t.

2. $b^2=a^2+c^2+2a'c$. Solve for c.

3. $x=\dfrac{-b\pm\sqrt{b^2+4ac.}}{2a}$ Solve for b.

4. $V=\frac{1}{3}a(b+b'+\sqrt{bb'})$. Solve for b.

5. $x=\sqrt{r(2r-\sqrt{4r^2-s^2})}$. Solve for r.

6. $x=\sqrt{r(2r-\sqrt{4r^2-s^2})}$. Solve for s.

CHAPTER XIII

SIMULTANEOUS EQUATIONS

170 In problems where several things are to be found, it is often convenient to make use of more than one unknown. For example: the sum of two numbers is 29 and one of them is 8 more than $\frac{3}{4}$ of the other. Find the numbers.

Let x = one number

y = the other number.

$$\text{then} \begin{cases} x+y = 29 \\ x = \frac{3}{4}y+8. \end{cases}$$

171 If a series of equations containing two or more unknowns can be checked by the same set of values for the unknowns, they are called *simultaneous equations.*

172 If *two* simultaneous equations can be combined in such a way as to produce *one* equation containing one unknown, they can be solved. This has already been done in solving formulas (Art. 125). For example:

$$\begin{cases} F = 1\frac{1}{2}d + \frac{1}{8} & (1) \\ d = 1\frac{5}{8} & (2) \end{cases}$$

$$F = 1\frac{1}{2} \cdot 1\frac{5}{8} + \frac{1}{8}$$

$$F = \frac{39}{16} + \frac{1}{8}$$

$$F = \frac{41}{16}$$

Observe that by substituting the value of the unknown in equation (2) for that unknown in (1), an equation containing only *one* unknown is obtained.

This same method can be applied to problems where the *actual* value of one unknown is not given.

Example 1: Solve: $\begin{cases} 2x=3y+5 & (1) \\ y=x-3 & (2) \end{cases}$

$$2x=3(x-3)+5 \text{ (substituting } x-3 \text{ for } y \text{ in (1))}$$
$$2x=3x-9+5$$
$$x=4$$
$$y=x-3=4-3=1$$

173 This method of solving simultaneous equations is called *substitution.*

Example 2: Solve: $\begin{cases} \dfrac{x+y}{x-y}=4-\dfrac{9}{x-y} & (1) \\ 4x-3y=23 & (2) \end{cases}$

$$\begin{cases} 5y-3x=-9 \text{ (simplifying (1))} & (3) \\ 4x-3y=23 & (2) \end{cases}$$

$$y=\frac{3x-9}{5} \text{ (solving (3) for y)}$$

$$4x-3\left(\frac{3x-9}{5}\right)=23 \text{ (substituting } \frac{3x-9}{5} \text{ for y in (2))}$$

$$20x-9x+27=115 \text{ (clearing of fractions).}$$
$$11x=88 \quad \text{Why?}$$
$$x=8 \quad \text{Why?}$$
$$y=\frac{3x-9}{5}=\frac{24-9}{5}=3$$

Check:

Check in equation (1): $\begin{cases} \dfrac{8+3}{8-3}=4-\dfrac{9}{8-3} \\ \dfrac{11}{5}=4-\dfrac{9}{5} \\ \dfrac{11}{5}=\dfrac{11}{5} \end{cases}$

Check in equation (2): $\begin{cases} 4\cdot 8-3\cdot 3=23 \\ 32-9=23 \\ 23=23 \end{cases}$

Example 3: Solve: $\begin{cases} 3x = 2y + 22 & (1) \\ \dfrac{x}{4} = \dfrac{10}{y} & (2) \end{cases}$

$\begin{cases} 3x = 2y + 22 \\ xy = 40 \text{ (simplifying (2))} & (3) \end{cases}$

$$x = \frac{40}{y} \quad \text{(solving (3) for x)}$$

$$3\left(\frac{40}{y}\right) = 2y + 22 \quad \left(\text{substituting } \frac{40}{y} \text{ for x in (1) }\right)$$

$$120 = 2y^2 + 22y \qquad \text{Why?}$$

$$y^2 + 11y - 60 = 0 \qquad \text{Why?}$$

$$(y+15)(y-4) = 0$$

$$y = -15 \quad \text{or } +4$$

$$x = \frac{40}{y} = \frac{40}{-15} \text{ or } \frac{40}{+4}$$

$$x = -\tfrac{8}{3} \quad \text{or } +10$$

$$\therefore x = -\tfrac{8}{3},\ y = -15 \quad \text{or } x = +10,\ y = +4$$

Check:

Check in equation (1) for $x = -\frac{8}{3}$, $y = -15$ $\begin{cases} 3(-\frac{8}{3}) = 2(-15) + 22 \\ -8 = -30 + 22 \\ -8 = -8 \end{cases}$

Check in equation (2) for $x = -\frac{8}{3}$, $y = -15$ $\begin{cases} \dfrac{-\frac{8}{3}}{4} = \dfrac{10}{-15} \\ -\frac{2}{3} = -\frac{2}{3} \end{cases}$

Check in equation (1) for $x = +10$, $y = +4$ $\begin{cases} 3\ (10) = 2 \cdot 4 + 22 \\ 30 = 8 + 22 \\ 30 = 30 \end{cases}$

Check in equation (2) for $x = +10$, $y = +4$ $\begin{cases} \dfrac{10}{4} = \dfrac{10}{4} \end{cases}$

Observe:

I. In checking simultaneous equations, the values must check in each equation.

II. When each of the unknowns has more than one value, the *corresponding pairs* of values must be used in checking. In Example 3, $x=-\frac{8}{3}$, $y=4$ would *not* check.

174 **RULE: Simplify each equation.**

Solve one of the equations for the unknowns, and substitute the value obtained for that unknown in the other equation.

Exercise 126

Solve and check:

1. $\begin{cases} 2x+3y=26 \\ x-5y=0 \end{cases}$

2. $\begin{cases} 3y=5x-18 \\ y=\dfrac{2x}{3} \end{cases}$

3. $\begin{cases} m=2n+3 \\ m=7n-12 \end{cases}$

4. $\begin{cases} \dfrac{2x+y}{4x-y}=3+\dfrac{4}{4x-y} \\ \dfrac{x}{y}=\dfrac{1}{3} \end{cases}$

5. $\begin{cases} \dfrac{r}{s}=2 \\ \dfrac{r}{10+s}=\frac{2}{3} \end{cases}$

6. $\begin{cases} 2x+\dfrac{5}{y}=\dfrac{89}{6} \\ 4x-27\frac{1}{2}=\dfrac{3}{y} \end{cases}$

7. $\begin{cases} x+y=15 \\ xy=56 \end{cases}$

8. $\begin{cases} \dfrac{1}{a}+\dfrac{1}{b}=\dfrac{7}{3} \\ a+b=\dfrac{7}{2} \end{cases}$

9. $\begin{cases} \dfrac{p+q-2}{p-q}=-\dfrac{1}{3} \\ \dfrac{3p+q-3}{2q-p}=-\dfrac{1}{11} \end{cases}$

10. $\begin{cases} \dfrac{2}{x+y}+\dfrac{26}{x-y}=\dfrac{7x-y-3}{x^2-y^2} \\ \dfrac{\frac{x+2}{y+9}}{\frac{x+8}{y+11}}=1 \end{cases}$

Exercise 127

Solve and check:

1. $\begin{cases} \dfrac{x}{y}=\dfrac{4}{3} \\ \dfrac{x}{65-y}=-\dfrac{1}{17} \end{cases}$

2. $\begin{cases} \dfrac{x^2+y^2}{4}=\dfrac{2}{9} \\ x+y=0 \end{cases}$

3. $\begin{cases} x^2+y^2=\dfrac{13}{36} \\ x-y=\dfrac{1}{6} \end{cases}$

4. $\begin{cases} x^2-xy+y^2=3 \\ x-y=1 \end{cases}$

5. $\begin{cases} xy+3=10 \\ x^2+y^2-10=0 \end{cases}$

6. $\begin{cases} x^2-y^2-9=0 \\ xy-12=0 \end{cases}$

7. $\begin{cases} x^2+y^2=90000 \\ x=y\sqrt{3} \end{cases}$

8. $\begin{cases} \sqrt{x^2-21}=-2y \\ y=x-6 \end{cases}$

9. $\begin{cases} \dfrac{2}{x}=y \\ \dfrac{x}{y}+\dfrac{y}{x}=\dfrac{65}{8} \end{cases}$

10. $\begin{cases} \dfrac{x}{6}=\dfrac{y^3}{7y^2-x^2} \\ 2x-1=y \end{cases}$

Exercise 128

Solve and check:

1. $\begin{cases} x+y=s \\ x-y=d \end{cases}$

2. $\begin{cases} \dfrac{x}{a+1}=\dfrac{a-1}{y} \\ \dfrac{x+y}{2}=a \end{cases}$

3. $\begin{cases} mg-mf=T \\ T=nf \end{cases}$ (Solve for T and f.)

4. $\begin{cases} \dfrac{x^2+y^2}{m^2+1}=13 \\ x+1=5m-y \end{cases}$

5. $\begin{cases} sx-r^2=rs-ry \\ x-r=s+y \end{cases}$

6. $\begin{cases} \dfrac{x-y}{m^2-n^2}=2 \\ \dfrac{x}{m}-3n=3m-\dfrac{y}{n} \end{cases}$

7. $\begin{cases} \dfrac{y}{a}=\dfrac{2b}{x-a} \\[2ex] \dfrac{x}{b}=\dfrac{2a}{y-b} \end{cases}$

9. $\begin{cases} ay=ac+x(b-c) \\ a-c)y=b(x-c)+a(a-x) \end{cases}$

8. $\begin{cases} \dfrac{2x-b}{a}=\dfrac{3x-y}{a+2b} \\[2ex] \dfrac{2x-b}{a}=\dfrac{a-2y}{b} \end{cases}$

10. $\begin{cases} \dfrac{m}{x}-c=\dfrac{n}{y} \\[2ex] \dfrac{n}{x}=\dfrac{m}{y}+c \end{cases}$

Special Cases of Simultaneous Equations

175 The four principles used in solving equations containing one unknown are:

1. *If equals are added to equals, the results are equal.*
2. *If equals are subtracted from equals, the results are equal.*
3. *If equals are multiplied by equals, the results are equal.*
4. *If equals are divided by equals, the results are equal.* (See Art. 15.)

These principles can be used also in combining simultaneous equations. For example:

$$\begin{cases} 5x+3y=17 \\ 4x-3y=10 \end{cases}$$
$$9x=27 \quad \text{(Principle 1)}$$
$$x=3$$

$$\begin{cases} \dfrac{x-1}{x+y}=2 \\[2ex] \dfrac{x+y}{x}=1 \end{cases}$$
$$\frac{x-1}{x}=2 \quad \text{(Principle 3)}$$

$$\begin{cases} 5x+3y=17 \\ 4x+3y=8 \end{cases}$$
$$x=9 \quad \text{(Principle 2)}$$

$$\begin{cases} x^2-y^2=8 \\ x-y=4 \end{cases}$$
$$x+y=2 \quad \text{(Principle 4)}$$

Observe that these principles may be used:

I. To cause one unknown to disappear (to eliminate one unknown) as in illustrations 1, 2 and 3.

II. To produce a set of equations of simpler form as in illustration 4. $\begin{cases} x+y=2 \\ x-y=4 \end{cases}$ are more simple to deal with than $\begin{cases} x^2-y^2=8 \\ x-y=2 \end{cases}$

Example 1: Solve: $\begin{cases} 3x-8y=19 & (1) \\ 5x+6y=-7 & (2) \end{cases}$

$$\begin{array}{rll} 9x-24y=57 & \text{(Multiplying (1) by 3)} & (3) \\ \underline{20x+24y=-28} & \text{(Multiplying (2) by 4)} & (4) \\ 29x=29 & \text{(Adding (3) and (4))} & \\ x=1 & & \\ 3\cdot 1-8y=19 & \text{(Substituting } x=1 \text{ in (1))} & \\ -8y=16 & & \\ y=-2 & & \end{array}$$

$$\therefore x=1,\ y=-2.$$

Example 1 may be solved as follows:

$$\begin{array}{rll} \begin{cases} 3x-8y=19 \\ 5x+6y=-7 \end{cases} & & \begin{array}{c}(1)\\(2)\end{array} \\ 15x-40y=95 & \text{(Multiplying (1) by 5)} & (3) \\ \underline{15x+18y=-21} & \text{(Multiplying (2) by 3)} & (4) \\ -58y=116 & \text{(Subtracting (4) from (3))} & \\ y=-2 & & \\ 5x+6(-2)=-7 & \text{(Substituting } y=-2 \text{ in (2))} & \\ 5x-12=-7 & & \\ 5x=\ \ 5 & & \\ x=\ \ 1 & & \end{array}$$

Check in (1) $\begin{cases} 3\cdot 1-8(-2)=19 \\ 3+16=19 \\ 19=19 \end{cases}$ Check in (2) $\begin{cases} 5\cdot 1+6(-2)=-7 \\ 5-12=-7 \\ -7=-7 \end{cases}$

Observe:

I. In order to eliminate one unknown, it may be necessary to multiply the equations by such numbers as will make the coefficients of that unknown the same.

II. *Either* unknown may be eliminated.

III. After one unknown is found, the other is obtained by substituting the value of the one found in *either* of the given equations.

176 This method of solving simultaneous equations is called *Addition* or *Subtraction.*

Example 2: Solve: $\begin{cases} x+3y=-1\frac{1}{2} & (1) \\ x-4\frac{1}{2}=6y & (2) \end{cases}$

$$x=-1\tfrac{1}{2}-3y \quad \text{(Solving (1) for x)}$$
$$x=6y+4\tfrac{1}{2} \quad \text{(Solving (2) for x)}$$
$$-1\tfrac{1}{2}-3y=6y+4\tfrac{1}{2} \quad \text{(Each member = x)}$$
$$-6=9y \quad \text{Why?}$$
$$y=-\tfrac{2}{3} \quad \text{Why?}$$
$$x=6(-\tfrac{2}{3})+4\tfrac{1}{2}=-4+4\tfrac{1}{2}=\tfrac{1}{2}.$$
$$\therefore x=\tfrac{1}{2},\ y=-\tfrac{2}{3}$$

Check in (1) $\begin{cases} \frac{1}{2}+3\,(-\frac{2}{3})=-1\frac{1}{2} \\ \frac{1}{2}-2=-1\frac{1}{2} \\ -1\frac{1}{2}=-1\frac{1}{2} \end{cases}$

Check in (2) $\begin{cases} \frac{1}{2}-4\frac{1}{2}=6\,(-\frac{2}{3}) \\ \frac{1}{2}-4\frac{1}{2}=-4 \\ -4=-4 \end{cases}$

Observe that $-1\frac{1}{2}-3y$ and $4\frac{1}{2}+6y$ are equal to each other because they are each equal to x. This principle may be stated as follows:

If two quantities are equal to the same thing, they are equal to each other.

177 This method of solving simultaneous equations is called *Comparison.*

Example 3: Solve: $\begin{cases} x^3-y^3=127 & (1) \\ x-y=1 & (2) \end{cases}$

$$\begin{cases} x^2+xy+y^2=127 \quad \text{(Dividing (1) by (2))} & (3) \\ x-y=1 & (2) \end{cases}$$

Dividing equation (1) by (2) gives an equation which is simpler to deal with than (1). Equations (3) and (2) may be solved by the method of substitution. The following method also may be used in equations of certain types.

$$\begin{array}{rll} x^2+xy+y^2= & 127 & & (3) \\ \underline{x^2-2xy+y^2=} & \underline{1} & \text{(Squaring (2))} & (4) \\ 3xy= & 126 & \text{(Subtracting (4) from (3))} & \\ xy= & 42 & & (5) \\ x^2+2xy+y^2= & 169 & \text{(Adding (3) and (5))} & (6) \\ x+y= & \pm 13 & \text{Why?} & (7) \\ \underline{x-y=} & \underline{1} & & (2) \\ 2x= & 14 \text{ or } -12 & & \\ x= & 7 \text{ or } -6 & & \\ y= & 6 \text{ or } -7 & & \end{array}$$

$\therefore x=7, \ y=6$

$x=-6, y=-7$

Check in (1) $\quad \begin{cases} 343-216=127 \\ 127=127 \end{cases}$
$x=7, y=6$

Check in (2) $\quad \begin{cases} 7-6=1 \\ 1=1 \end{cases}$
$x=7, y=6$

Check in (1) $x=-6$, $y=-7$
$$\begin{cases}(-216)-(-343)=127\\ -216+343=127\\ 127=127\end{cases}$$

Check in (2) $x=-6$, $y=-7$
$$\begin{cases}(-6)-(-7)=1\\ -6+7=1\\ 1=1\end{cases}$$

Observe:

I. In order to obtain equation (7), a trinomial square was necessary.

II. In order to obtain the trinomial square, equation (5) was needed to combine with (3).

III. This method will apply only to problems where it is possible to obtain one or more equations in the form of a trinomial square.

Exercise 129

Solve and check: (Select the method which seems best suited to the problem.)

1. $\begin{cases}x+3y=13\\ x=7y-37\end{cases}$

2. $\begin{cases}8x^3-y^3=271\\ 2x-y=1\end{cases}$

3. $\begin{cases}x^3-27y^3=35\\ x^2+3xy+9y^2=7\end{cases}$

4. $\begin{cases}3s-8t=-17\\ 7s+6t=-15\end{cases}$

5. $\begin{cases}3x^2+4y^2=\dfrac{7}{12}\\ 4x^2-5y^2=\dfrac{19}{144}\end{cases}$

6. $\begin{cases}x^2-6xy+8y^2=-\dfrac{5}{4}\\ x-4y=-\dfrac{5}{2}\end{cases}$

7. $\begin{cases}4x+y=-\dfrac{3}{7}\\ y-2x=+\dfrac{3}{7}\end{cases}$

8. $\begin{cases}6x^2+8y^2=54\\ 9x^2-12y^2=9\end{cases}$

9. $\begin{cases} y-2z= 1 \\ y^3+8z^3=189. \end{cases}$ Check for the real value only

10. $\begin{cases} x^2+xy+y^2=37 \\ x^2-xy+y^2=13 \end{cases}$

Problems Involving Simultaneous Equations

Example 1: The difference between two sides of a right triangle is 7, and the hypotenuse is 13. Find the sides.

Let x = one side.

Let y = the other side.

then $\begin{cases} x-y= 7 & (1) \\ x^2+y^2=169 & (2) \end{cases}$

Solving $x=12,\quad y=5$

$x=-5,\ y=-12$

Check $x=12,\ y=5$ $\begin{cases} 12-5= 7 & \text{in (1)} \\ 144+25=169 & \text{in (2)} \end{cases}$

Check $x=-5,\ y=-12$ $\begin{cases} -5-(-12)= 7 & \text{in (1)} \\ 25+144=169 & \text{in (2)} \end{cases}$

NOTE: In problems where the negative values have no meaning, they may be discarded.

Example 2: The ratio of two numbers is $\frac{3}{2}$ and the sum of the numbers is 8 more than 4 times their difference. Find the numbers.

Let x = one number

y = the other

then $\frac{x}{y}=\frac{3}{2}$

$\underline{x+y=4(x-y)+8}$

Solving $x=24.\quad y=16.$

Check: $\begin{cases} \frac{24}{16}=\frac{3}{2} \text{ in (1)} \\ 24+16=4\ (24-16)+8 \text{ in (2)} \end{cases}$

Exercise 130

1. The sum of two angles is 66°, and their difference is 16°. Find the angles.

2. The ratio of two angles is $\frac{8}{7}$, and the ratio of their complements is $\frac{10}{11}$. Find the angles.

3. The perimeter of a rectangle is 28, and the diagonal is 10. Find the length and width.

4. The perimeter of one square is 6″ more than that of another, and the sum of their areas is 37.25 sq. in. Find the sides of the squares.

5. The R. P. M. of two pulleys belted together, are respectively 630 and 840, and one is 7″ larger than the other. Find the size of each pulley.

6. Two angles are in the ratio $\frac{7}{8}$, and they are supplementary. Find them.

7. The perimeter of a rectangle is 46, and the area is 120. Find the dimensions.

8. The area of a rectangle is 360, and the diagonal is 41. Find the dimensions.

9. The R. P. M. of two meshed gears are respectively 560 and 720, one has 8 more teeth than the other. How many teeth has each?

10. The speed of the motor on a lathe is 1200 R. P. M., and the speed of the lathe when belted for low speed is 600 R. P. M., and when belted for high speed is 4200 R. P. M. The difference in diameters of the steps is 2″. Find the diameters of the three steps on both pulleys.

11. The difference of two numbers is $\frac{1}{5}$, and the difference of their squares is .$\frac{4}{5}$ Find the numbers.

12. The sum of two numbers is 1, and the sum of their cubes is $\frac{1}{3}$. Find the numbers.

13. The difference of two numbers is 2, and the difference of their cubes is 2. Find the numbers.

14. A rectangular lot is surrounded by a walk 6 ft. wide. The area of the lot is 555 sq. yds., and of the walk is 224 sq. yds. Find the length and width of the lot.

15. The difference of the areas of two squares is $2\frac{1}{2}$, and the ratio of the areas is $\frac{49}{9}$. Find the side of each square.

Beam Problems

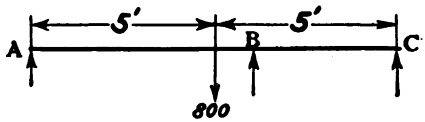


Fig. 78

178 Suppose an I-beam 10 ft. long and weighing 800 lbs. is supported at three points, A, B and C, as shown in Fig. 78. If the supports at A and B are removed, it is evident that the fulcrum is at C. If the supports at A and C are removed, the fulcrum is at B, and if the supports at B and C are removed, the fulcrum is at A. If then there is more than one actual support about which the lever could turn if not in balance, any of these supports may be used for a fulcrum.

179 Beam. A beam is a lever which has *no one* point about which it must turn.

Example: Two weights 640 lbs. and 360 lbs. are attached to a beam supported at the ends (Fig. 79). The supports at the ends carry loads which are in the ratio $\frac{2}{3}$. Find the loads.

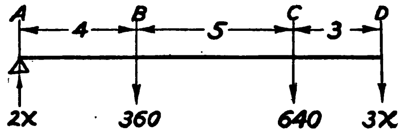


Fig. 79

1. Place the fulcrum at A. (Fig. 79.) Then (Art. 77)

$$(+2x)(0)+(-360)(+4)+(-640)(+9)+(+3x)(+12)=0$$
$$0 \quad - \quad 1440 \quad -5760 \quad + \quad 36x=0$$
$$36x=7200$$
$$x=200$$
$$2x=400$$
$$3x=600$$

2. Place the fulcrum at B. (Fig. 79). Then (Art. 77)

$$(+2x)(-4)+(-360)(0)+(-640)(+5)+(+3x)(+8)=0$$
$$-8x \quad +0 \quad - \quad 3200 \quad +24x=0$$
$$16x=3200$$
$$x=200$$
$$2x=400$$
$$3x=600$$

3. Place the fulcrum at C. (Fig. 79.) Then (Art. 77)

$$(+2x)(-9)+(-360)(-5)+(-640)(0)+(+3x)(+3)=0$$
$$-18x \quad + \quad 1800 \quad +0 \quad +9x=0$$
$$-9x=-1800$$
$$x=200$$
$$2x=400$$
$$3x=600$$

4. Place the fulcrum at D. (Fig. 79.) Then (Art. 77)

$$(+2x)(-12)+(-360)(-8)+(-640)(-3)+(+3x)(0)=0$$
$$-24x \quad + \quad 2880 \quad + \quad 1920 \quad +0=0$$
$$-24x=-4800$$
$$x=200$$
$$2x=400$$
$$3x=600$$

Observe:

I. The same *result* but *not* the same equation is obtained for each position of the fulcrum.

II. The leverage of any force at the fulcrum becomes zero.

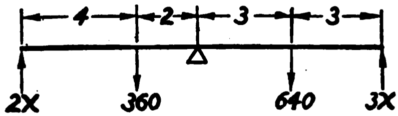


Fig. 80

5. Place the fulcrum at a point where there is no force applied, as at the middle point. (Fig. 80.) Then

$$(+2x)(-6)+(-360)(-2)+(-640)(+3)+(+3x)(+6)=0$$

(Art. 77)

$$-12x \quad + \quad 720 \quad -1920 \quad +18x=0$$
$$6x=1200$$
$$x=200$$
$$2x=400$$
$$3x=600$$

Observe:

I. The same result is again obtained. Hence,

II. *The fulcrum may be placed at any point.*

180 Beam problems often arise in which two or more of the forces or arms are unknown, making it convenient to use simultaneous equations. In that case one equation is obtained by applying the *Law of Leverages* (Art. 77), and the other by applying the *Law of Forces.*

Suppose two men are carrying a weight of 150 lbs. by means of a pole on their shoulders. It is easily seen that the upward forces together must equal the downward force. $x+y=150$. (Fig. 81).

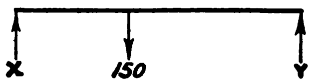


Fig. 81

181 Law of Forces. The sum of the forces acting upon a beam in one direction must equal the sum of the forces acting upon it in the opposite direction.

Example: Three men wish to carry a weight of 150 lbs. suspended from a pole 12 ft. long and weighing 60 lbs. One man lifts one end of the pole and the other two lift the other end by means of a cross bar placed under the pole 2 ft. from that end. If the weight is placed $2\frac{1}{2}$ ft. from the cross bar, and if the two men at the cross bar lift the same amount, find what each of the three men lift.

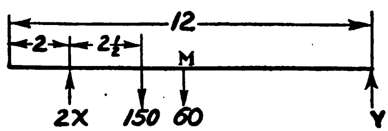


Fig. 82

Let x = amount lifted by each man at the cross bar.

Let y = amount lifted by the man at the other end.

Place the fulcrum at the middle (Fig. 82). Then

$$(+2x)(-4)+(-150)(-1\tfrac{1}{2})+(-60)(0)+(+y)(+6)=0 \qquad (1)$$

(Art. 77)

$$2x+y=150+60 \qquad \text{(Art. 181)} \quad (2)$$

$$\begin{cases} -8x+6y=-225 & \text{(Simplifying (1))} \\ 2x+\ y=\ \ 210 & \text{(Simplifying (2))} \end{cases}$$

Solving, $x=74.25$

$y=61.5$

Check in (1)

$$\begin{cases} (2\cdot 74.25(-4)+(-150)(-1\tfrac{1}{2})+(-60)(0)+(+61.5)(+6)=0 \\ -594 \quad + \quad 225 \quad + \quad 0 \quad + \quad 369 \quad =0 \\ -594 \quad + \quad 594 \quad =0 \end{cases}$$

Check in (2) $\begin{cases} 2\cdot 74.25+61.5=210 \\ 148.5+61.5=210 \\ 210=210 \end{cases}$

Exercise 131

1—5. Find the unknown forces and arms in the following figures:

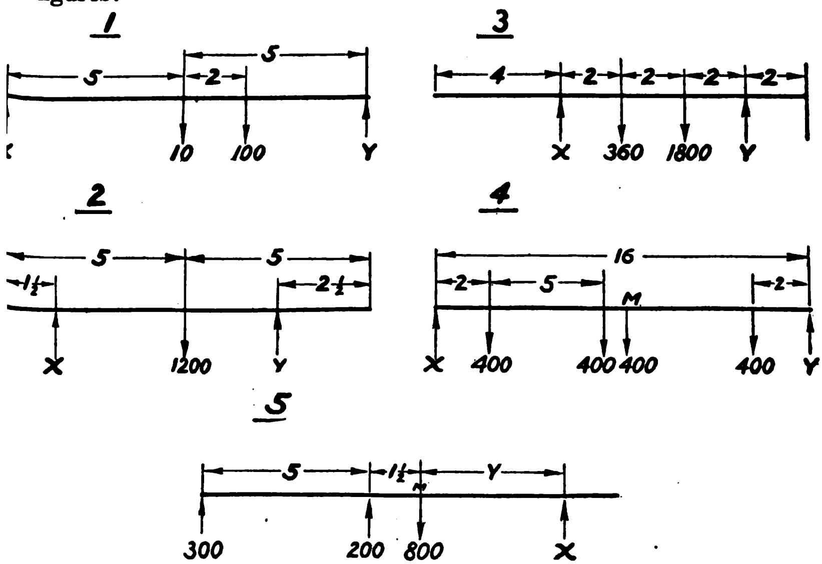


Fig. 83

6. A wooden beam 16 ft. long and weighing 420 lbs., carries a load of 3000 lbs. 5 ft. from one end. Find the pressure on the support at each end of the beam.

7. A bridge is 24 ft. long and weighs 5 tons. What weight is borne by each support at the ends when a wagon weighing 2500 lbs. is one third the way across?

8. A man and a boy are to carry 400 lbs. on a pole 9 ft. long. How far from the boy must the load be placed so that he shall carry one fourth of the weight?

9. A timber 16 ft. long weighing 1600 lbs. is supported by two iron posts placed 2 ft. from the ends. A weight of 2400 lbs. is resting upon the timber 3 ft. from the center. Find the pressure upon each post.

10. An I-beam 18 ft. long, weighing 30 lbs. per foot is being moved by wheels and axles placed 2′ and 4′ from the ends. What will be the weight on each axle?

11. A horizontal timber 24 ft. long has to support weights of 250 lbs. each, placed at the ends and at intervals of 6 ft. throughout the length. One support of the timber is 7 ft. from the center, and the other is 10 ft. from the center. What weight is carried by each support?

12. A timber $24'' \times 30'' \times 21'$ is being moved by a lumber wagon coupled to 10 ft. between the axles. The timber extends 4 ft. beyond the front axle. If the timber weighs 48 lbs. per cubic foot, how much weight will each axle carry?

13. A horizontal spring board 12 ft. long is fastened at one end, and at a point 2 ft. from that end. A man weighing 160 lbs. is standing on the other end. Find the forces exerted at the two stationary points.

14. Two men carrying a weight on a pole 16 ft. long, lift weights respectively of 100 and 140 lbs. What weight are they carrying, and how far from the second man is it placed?

15. A wagon loaded with 500 bricks weighing $5\frac{1}{2}$ lbs. apiece, is being drawn across a bridge. The wagon box is 10 ft. long, and extends 2 ft. beyond the front axle and $2\frac{1}{2}$ ft. beyond the rear axle. The bridge is 32 ft. long and weighs $5\frac{1}{2}$ tons. Find the pressure on the piers at the ends of the bridge when the front axle of the wagon is $\frac{5}{8}$ the way across.

Simultaneous Equations with Three or More Unknowns

182 To solve equations containing *two* unknown quantities, it was necessary to have *two* equations. To solve equations containing *three* unknown quantities, *three* equations are necessary. The method of addition or subtraction makes it possible to eliminate *one* unknown and to obtain *two* new equations each containing the *other two* unknowns.

Example 1: Solve: $\begin{cases} x+y+z=1 & (1) \\ 5x-3y-z=5 & (2) \\ 4x+y+2z=3 & (3) \end{cases}$

Eliminate y by combining (1) and (2) and also by combining (1) and (3).

$$\begin{array}{ll} 3x+3y+3z=3 \text{ (multiplying (1) by 3)} & (4) \\ \underline{5x-3y-z=5} & (2) \\ 8x+2z=8 \text{ (adding)} & \end{array}$$

$$\begin{array}{ll} x+y+z=1 & (1) \\ \underline{4x+y+2z=3} & (3) \\ 3x+z=2 \text{ (subtracting (1) from (3))} & (5) \end{array}$$

$$\begin{array}{ll} 8x+2z=8 & (4) \\ \underline{3x+z=2} & (5) \end{array}$$

Solving (4) and (5) by *any* method preferred, $x=2$, $z=-4$.

To find y, substitute $x=2$, $z=-4$, in *any* of the origina equations.

$$2+y-4=1 \quad \text{(substituting in (1))}$$
$$y=3$$

Check: $\begin{cases} 2+3+(-4) = 1 & \text{in (1)} \\ 10-9-(-4) = 5 & \text{in (2)} \\ 8+3+2(-4) = 3 & \text{in (3)} \end{cases}$

Example 2: Solve: $\begin{cases} x+3y+z=1 & (1) \\ 2x-6y \quad =0 & (2) \\ \dfrac{x}{3}-y-2z=6 & (3) \end{cases}$

$$\begin{array}{lr} x+3y+z=1 & (1) \\ 2x-6y \quad =0 & (2) \\ \underline{x-3y-6z=18} \text{ (clearing (3) of fractions)} & (4) \end{array}$$

Eliminate z in (1) and (4).

$$\begin{array}{lr} 6x+18y+6z=\ 6 \text{ (multiplying (1) by 6)} & \\ \underline{\ x-\ 3y-6z=18} & \\ 7x+15y \quad\ =24 \text{ (adding)} & (5) \end{array}$$

$$\begin{cases} 2x-6y \ =0 & (2) \\ 7x+15y =24 & (5) \end{cases}$$

Solving (2) and (5), $x=2$, $y=\frac{2}{3}$.

$$2+3(\tfrac{2}{3})+z=1 \quad \text{(substituting } x=2,\ y=\tfrac{2}{3} \text{ in (1))}$$
$$z=-3$$

Check: $\begin{cases} 2+3(\frac{2}{3})+(-3) = 1 \text{ in (1)} \\ 2\cdot2-6\cdot\frac{2}{3} \quad = 0 \text{ in (2)} \\ \frac{2}{3}-\frac{2}{3}-2(-3) = 6 \text{ in (3)} \end{cases}$

Exercise 132

Solve and check:

1. $\begin{cases} 2x+2y+3z=4 \\ 3x+4y+6z=7 \\ x+2y+6z=4 \end{cases}$

2. $\begin{cases} x-y-2z=2 \\ x+y-2z=16 \\ 5x+\dfrac{y}{14}+\dfrac{z}{2}=3\frac{1}{2} \end{cases}$

3. $\begin{cases} x+y+2z=\frac{1}{3} \\ 2x+3y-12z=4 \\ 6(x+y)=4-6z \end{cases}$

4. $\begin{cases} x+y+z=6a \\ 2x-y+2z=6a \\ 7x-2y-3z=-6a \end{cases}$

5. $\begin{cases} 2x+y-z=7 \\ x-y=-1 \\ y-z=-1 \end{cases}$

CHAPTER XIV

THE GRAPH

183 *Co-ordinates:* Certain facts may be represented by means of drawings. For example: if a building in a city is described as being five blocks east of the main north and south street, and two blocks north of the main east and west street, the building can be exactly located on the city map. (Fig 84.) A point can be located if its distances and directions from two intersecting straight lines are known. The two intersecting lines are called *reference lines* and the two distances, the *co-ordinates* of the point. The horizontal distance is called the *abscissa* and is usually represented by x, and the vertical distance is called the *ordinate* and is usually represented by y. The directions are indicated by the plus and minus signs as in Art. 67. Unless otherwise stated the reference lines are at right angles to each other and paper ruled as in Fig. 85 may be used conveniently.

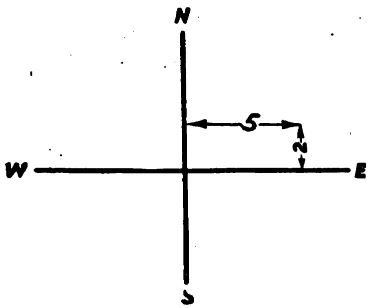


Fig. 84

A point is represented by the symbol (x, y). In Fig. 85, the point A is represented by (+5, +2), the point B by (−2, +5), the point C by (−3, −7), and the point D by (+4, −4). Observe that the abscissa is always given first.

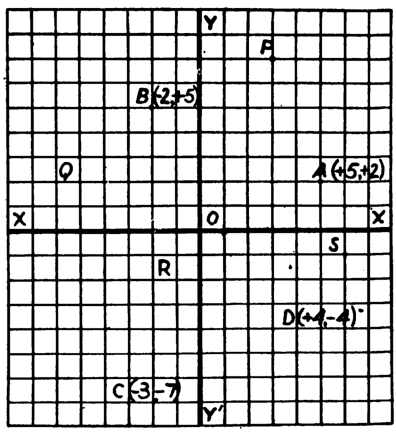


Fig. 85

How are the points P, Q, R, and S represented?

184 Location of Points: To locate a point, draw the reference lines, measure *horizontally* from their intersection the distance represented by the *abscissa* in the direction indicated by its sign; then measure *vertically* from that point the distance represented by the ordinate in the direction indicated by its sign.

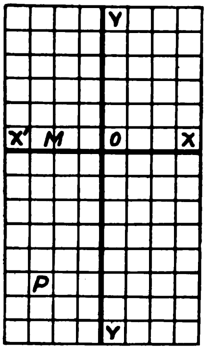


Fig. 86

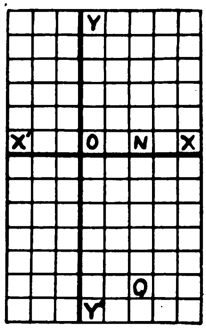


Fig. 87

Example 1: Locate the point (−2, −6). Draw the reference lines xx′, yy′(Fig. 86.) From their intersection O, measure distance equal to 2 toward the *left*, OM. From M, measure a distance equal to 6 *down*, MP. P is the point (−2, −6).

Example 2: Locate the point (+2, −6). Draw the reference lines xx′, yy′, (Fig. 87). From their point of intersection, O, measure a distance equal to 2 toward the *right*, ON. From N, measure a distance equal to 6 *down*, NQ. Q is the point (+2, −6).

Exercise 133

Locate the following points:

1. $(3, 1)$
2. $(1, 3)$
3. $(3, -1)$
4. $(-1, -3)$
5. $(-1, +3)$
6. $(-3, -1)$
7. $(4, 0)$
8. $(-4, 0)$
9. $(0, 8)$
10. $(0, -8)$
11. $(0, 0)$
12. $(2, \frac{1}{2})$
13. $(2\frac{1}{2}, -3\frac{1}{2})$
14. $(-1, \sqrt{3})$
15. $(-\frac{1}{2}\sqrt{2}, -\sqrt{53})$
16. $(\frac{1}{3}\sqrt{3}, \sqrt{5})$
17. $(6\sqrt{2}, -\frac{1}{4}\sqrt{5}$

18. Draw a triangle whose vertices are the points $(3, 5)$, $(-5, 1)$, and $(2, -3)$.

19. Draw a quadrilateral whose vertices are the points $(2, 2)$, $(2, -2)$, $(-5, -5)$, and $(-5, +5)$.

20. Draw a rectangle whose vertices are $(+7, +2)$, $(-3, +2)$, $(-3, -5)$, $(+7, -5)$, and find its area.

185 Graph: A *graph* is a figure representing the relation between quantities. The relation between x and y expressed in the formula $y = 3x - 2$, may be expressed by a graph as follows:

Evaluate the formula for several values of x, chosen at random (as in table). Using the corresponding values of x and y as co-ordinates, locate the points $(3, 7)$, $(0, -2)$, $(-2, -8)$, $(-3, -11)$. (Fig. 88). Draw a line through these points. This line is the *graph* of $y = 3x - 2$.

x	3	0	-2	-3
y	7	-2	-8	-11

Exercise 134

Graph:

1. $y=4x$
2. $y=2x+1$
3. $y=3x+5$
4. $y=2x-3$
5. $y=4x-1$

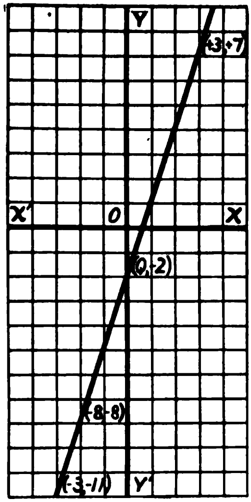


Fig. 88

Notice that these graphs seem to be straight lines. If other values of x are chosen in evaluating $y=3x-2$ (Fig. 88), and other points are located in the same way, such as (4, 10), (1, 1), $(-1, -5)$, they lie in the same line. Also if the co-ordinates of any point in the line such as $(2\frac{1}{3}, 5)$ are substituted in the formula, they check the equation; and if the co-ordinates of any point *not* on the line are substituted, such as (3, 5), the equation does *not* check.

186 The graph of an equation containing only the first power of the unknown is a straight line. Hence,

187 An equation of the first power is called a *linear equation.*

188 Since a *straight* line can be drawn if any *two* of its points are known, it is necessary to locate only *two* points when graphing an equation of the first power.

NOTE: For the sake of accuracy in drawing, these points should not be close together, and this can be accomplished by choosing values of *x* which are not close together, such as $x=+5$, and $x=-8$.

Example: Graph the equation $3x-2y=5$.

Solve the equation for y.

$$y=\frac{3x-5}{2}$$

Evaluate $y=\frac{3x-5}{2}$ for two values of x.

If $x=-3$, $y=-7$

If $x=+7$, $y=+8$.

Locate the points $(-3, -7)$ and $(+7, +8)$ and draw straight line through them.

AB (Fig. 89) is the graph of $3x-2y=5$.

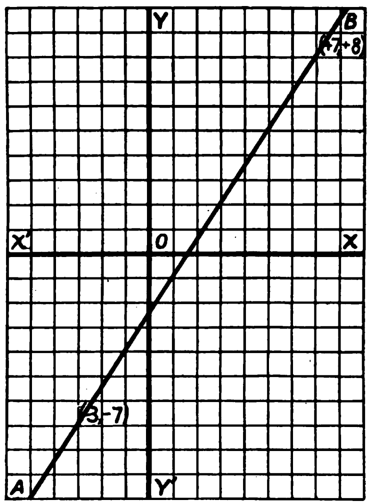


Fig. 89

Observe that for the sake of accuracy, it is better to choos values of x which will give *integral* values of y.

Exercise 135

Graph the equations:

1. $x+y=5$
2. $x=2y$
3. $2x-y=11$
4. $3x=5-y$
5. $7x+3y=10$
6. $3x=5y$
7. $5x=10y-2$
8. $x+y=0$
9. $x+y+3=0$
10. $9x+2y+1=0$

189 Let it be required to graph $y=2x^2+1$.

Evaluate the equation for several values of x.

x	3	2	1	0	−1	−2	−3
y	19	9	3	1	3	9	19

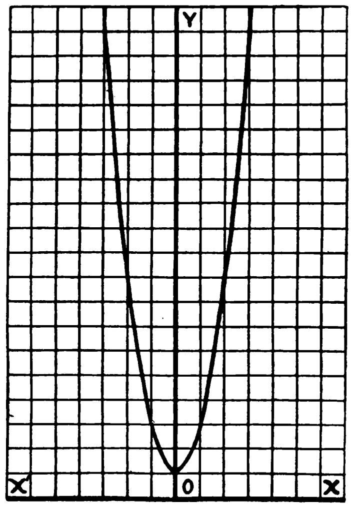


Fig. 90

Locate the points (3, 19), (2, 9), (1, 3), (0, 1), (−1, 3), (−2, 9), (−3, 19), and draw a curve through these points (Fig. 90). The curve is the graph of $y=2x^2+1$.

Observe:

I. The graph of an equation containing a second power of the unknown is not a straight line.

II. When the graph is not a straight line, it is necessary to evaluate the given expression for *more* than *two* values of *x*. Enough points must be located to thoroughly determine the curve.

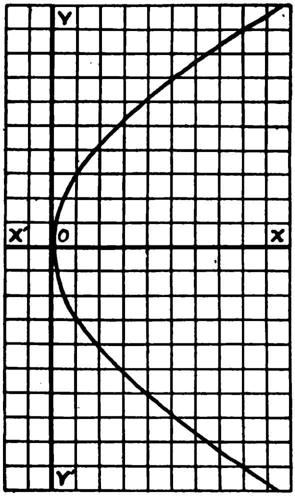


Fig. 91

190 Let it be required to graph $y^2 = 9x$.

Solve $y^2 = 9x$ for y.

$$y = \pm 3\sqrt{x}$$

Evaluate $y = \pm 3\sqrt{x}$ for several values of x.

x	0	1	2	4	8	9	12
y	0	±3	±4.24	±6	±8.48	±9	±10.39

Locate the points (0, 0), (1,+3,) (1,−3), (2,+4.24) (2,−4.24), (4,+6), (4,−6), (8,+8.48), (8,−8.48), (9,+9), etc.

Draw a curve through these points (Fig. 91). The curve is the graph of $y^2 = 9x$.

Observe that x can not have a negative value because it is impossible to find the square root of a negative number.

191 A curve which is the graph of an equation containing the square of only one unknown quantity is called a *Parabola* (Figs. 90, 91.)

Graph: **1.** $y = x^2$ **2.** $y^2 = x$

192 Let it be required to graph

$$x^2 + y^2 = 25.$$

Solve $x^2 + y^2 = 25$ for y.

$$y^2 = 25 - x^2$$

$$y = \pm\sqrt{25 - x^2}$$

Evaluate $y = \pm\sqrt{25 - x^2}$ for several values of x.

x	+5	+4	+3	+1	0	−1	−3	−4	−5
y	0	±3	±4	±4.9	±5	±4.9	±4	±3	0

Locate the points (5, 0), (+4, +3), (+4, −3, (+3, +4), (+3, −4), etc.

Draw a curve through these points (Fig 92). The curve is the graph of $x^2+y^2=25$.

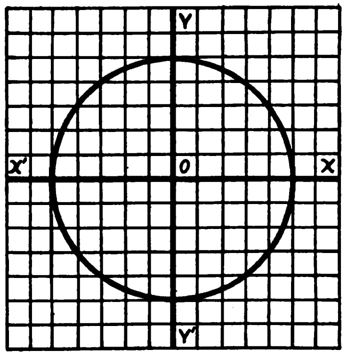


Fig. 92

193 Let it be required to graph $x^2+9y^2=36$.

Solve $x^2+9y^2=36$ for y.

$$9y^2=36-x^2$$

$$y^2=\frac{36-x^2}{9}$$

$$y=\frac{\pm\sqrt{36-x^2}}{3}$$

Evaluate $y=\frac{\pm\sqrt{36-x^2}}{3}$ for several values of x.

x	0	±2	±4	±6
y	±2	±1.89	±1.49	0

Locate the points (0, +2), (0, −2), (+2, +1.89), (+2, −1.89), (−2, +1.89), (−2, −1.89), etc.

Draw a curve through these points (Fig 93). The curve is the graph of $x^2+9y^2=36$.

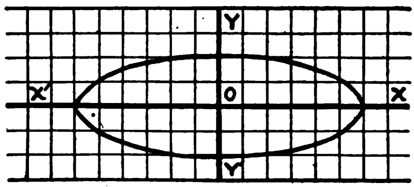


Fig. 93

194 A curve which is the graph of an equation containing the *positive* squares of *both* unknowns in the form $x^2+y^2=a^2$ is a *Circle* (Fig. 92).

195 A curve which is the graph of an equation of the form $x^2+ay^2=b$ is an *Ellipse* (Fig. 93).

Graph 1: $x^2+y^2=100$. 2. $9x^2+4y^2=36$.

196 Let it be required to graph $9y^2-4x^2=4$.

Solve $y^2-4x^2=4$ for y.

$$y^2=4x^2+4$$

$$y=\pm\sqrt{4x^2+4}$$

Evaluate $y=\pm\sqrt{4x^2+4}$ for several values of x.

x	0	±1	±2	±3	±4	±5
y	±2	±2.82	±4.47	±6.32	±8.25	±10.20

Locate the points (0, +2), (0, −2), (+1, +2.82), (+1, −2.82), (−1, +2.82), (−1, −2.82), (+2, +4.47), (+2, −4.47), etc.

Draw a curve through these points (Fig. 94). The curve is the graph of $y^2-4x^2=4$.

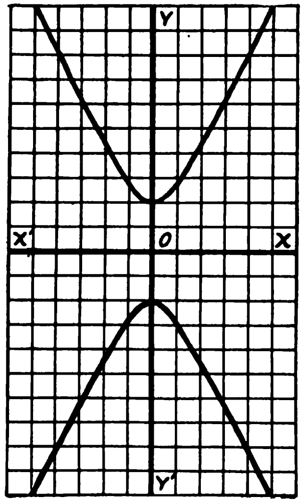


Fig. 94

197 A curve which is the graph of an equation of the form $x^2-ay^2=b$ is a *Hyberbola* (Fig. 94).

Graph $9x^2-4y^2=36$.

Exercise 136

Graph:

1. $x^2+y^2=16$
2. $x^2-y^2=16$
3. $9x^2+y^2=36$
4. $y^2=3x+1$
5. $4y=5x^2-20$
6. $y=x^2+2x-3$

7. $y^2+2y=4x+8$

Suggestion: Solve the equation for y.

$$y^2+2y+1=4x+9 \qquad \text{Why?}$$
$$y+1=\pm\sqrt{4x+9}$$
$$y=\pm\sqrt{4x+9}-1$$

Evaluate $y=\pm\sqrt{4x+9}-1$ for several values of **x**.

8. $x^2+y^2+2y=24$

9. $x^2-y^2+8y=0$

10. $x^2+4x+4y^2+16y=180$

Graphs of Simultaneous Equations

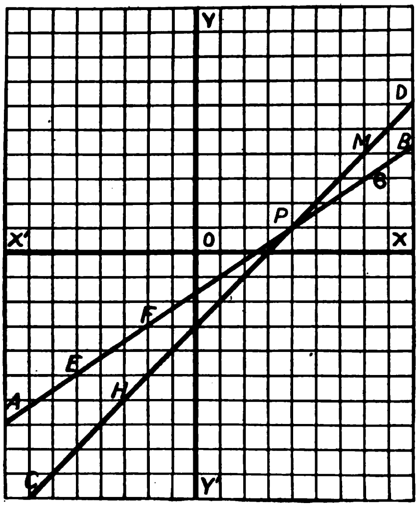


Fig. 95

198 The graph of $2x=3y+5$ is the line AB (Fig. 95). The graph of $y=x-3$ *using the same reference lines,* is the line CD (Fig. 95).

Observe:

I. The co-ordinates of the points E, F, P, G all check the equation $2x=3y+5$.

II. The co-ordinates of the points H, P, M all check the equation $y=x-3$.

III. The co-ordinates of the point P (4, 1) check both equations.

IV. Compare the co-ordinates of the point P with the values of x and y when these same equations were solved by substitution in Example 1, Art. 172.

199 If two equations are graphed using the same reference lines, the co-ordinates of each point of intersection check both equations, and are the values of the unknowns found when the equations are solved simultaneously.

Graphing furnishes a method of checking the number of solutions of simultaneous equations.

Example 1: Solve the simultaneous equations and check by graphing:

$$x^2+y^2=25 \qquad (1)$$

$$3x+4y=0 \qquad (2)$$

$$x^2+y^2=25 \qquad (1)$$

$$y=\frac{-3x}{4} \quad \text{Solving (2) for y}$$

$$x^2+\frac{9x^2}{16}=25$$

$$x^2=16$$

$$x=\pm4$$

$$y=\frac{-3x}{4}=\mp3$$

$\therefore$ $x=+4,\quad y=-3$

$x=-4,\quad y=+3$

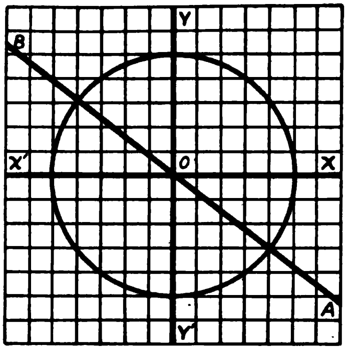


Fig. 96

Check: The graph of $x^2+y^2=25$ is the circle (Fig. 96), and the graph of $3x+4y=0$, *using the same reference lines*, is the line AB (Fig. 96).

The points of intersection are $(+4, -3)$ and $(-4, +3)$.

Compare results.

Example 2: Solve the equations and check by graphing:

$$3x^2+y^2=36 \qquad (1)$$

$$x^2-y^2=4 \qquad (2)$$

$$4x^2=40 \text{ (adding)}$$

$$x^2=10$$

$$= \pm\sqrt{10} = \pm 3.162$$

$$10-y^2=4$$

$$y^2=6$$

$$y=\pm 2.45$$

$$\therefore \quad x=3.162, \qquad y=\pm 2.45$$

$$x=-3.162, \qquad y=\pm 2.45$$

Check: The graph of $3x^2+y^2=36$ is the ellipse (Fig. 97). The graph of $x^2-y^2=4$ is the hyperbola (Fig. 97). The points of intersection are $(+3.16, +2.45)$, $(+3.16, -2.45)$, $(-3.16, +2.45)$, and $(-3.16, -2.45)$.

Compare results.

NOTE: The graph is only an approximate check on the accuracy of results.

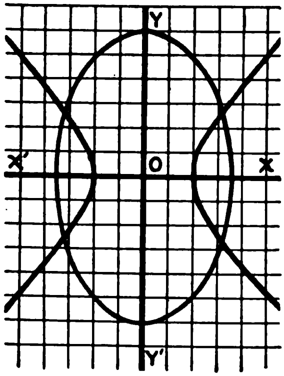


Fig. 97

Exercise 137

Solve and check by graphing:

1. $\begin{cases} x+y=4 \\ 6x+y=\frac{23}{2} \end{cases}$

2. $\begin{cases} 2x-3y=-3 \\ 3x-10y=1 \end{cases}$

3. $\begin{cases} y=x^2 \\ x=y-6 \end{cases}$

4. $\begin{cases} y^2=x+9 \\ x^2+y^2=9 \end{cases}$

5. $\begin{cases} 9x^2+25y^2=225 \\ 3x+5y=15 \end{cases}$

6. $\begin{cases} x^2+y^2=32 \\ x^2+3y^2=36 \end{cases}$

7. $\begin{cases} x^2-y^2+4=0 \\ x^2+3y^2=36 \end{cases}$

8. $\begin{cases} 3x^2+4y^2=48 \\ 4x^2+3y^2=48 \end{cases}$

9. $\begin{cases} x^2+y^2=9 \\ x^2+4y^2=36 \end{cases}$

10. $\begin{cases} x^2+y^2=16 \\ x^2-y^2=16 \end{cases}$

Graphing of Statistics

200 The graph is used to represent relations between quantities that are not always expressed by an equation. For example: suppose a thermometer is read every hour from 8 A. M. to 8 P. M. on January 3 and the readings are as follows:

	A.M.				Noon
Hour..................	8	9	10	11	12
Reading................	$-10°$	$-2°$	$+5°$	$+20°$	$+26°$

	P.M.							
Hour.........	1	2	3	4	5	6	7	8
Reading......	$+28°$	$+31°$	$+22°$	$+13°$	$+12°$	$+10°$	$+12°$	$+15°$

To graph the temperature for successive hours, the *time* is measured along a *horizontal* line, and the *temperature* along *vertical* lines. The number of spaces used to represent a unit of time and a unit of temperature may be chosen to suit the conditions of each problem. In this case, it is convenient to use one space for one hour horizontally, and one space for 5° vertically (Fig. 98).

A similar temperature graph for June 3 is given in Fig. 98.

At about what times was the temperature 60°? 75°? 78°?

What was the temperature at 12 noon? At 4 P. M.? At 11:30 A. M.?

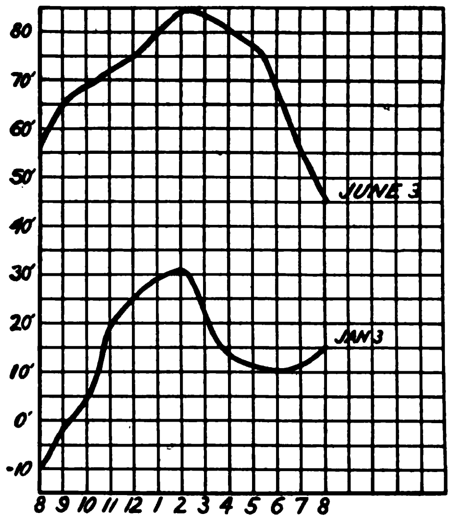


Fig. 98

Exercise 138

1. Graph the production of cotton in the U. S. from 1790 to 1910 from the statistics given in the following table:

Year	1790	1800	1810	1820	1830	1840
Millions of Bales	.1	.2	.25	.35	.7	1.8

Year	1850	1860	1870	1880	1890	1900	1910
Millions of Bales	2.1	3.8	4.1	6.3	8.7	10.3	12.1

2. The yearly output of an automobile factory for a period of years is given in the following table. Graph.

Year	1909	1910	1911	1912	1913	1914	1915	1916
Output	12,000	18,000	38,000	47,000	56,000	60,000	92,000	130,000

3. A student during one month of school work makes the following record. Graph his percentages.

No. of Problems Correct	7	6	18	6	2	1	10	0	8	5
No. of Problems Assigned	10	8	20	12	6	3	15	2	10	7
No. of Problems Correct	9	5	8	7	24	0	7	3	7	11
No. of Problems Assigned	9	8	10	8	25	6	14	5	7	11

4. A pulley is belted to a second pulley which is 21″ in diameter and has a speed of 240 R. P. M. Graph the change in speed, pulleys on the first shaft varying in size from 3″ to 24″.

Size of Pulley	3″	6″	9″	12″	15″	18″	21″	24″
R. P. M. of Pulley								

5. A line revolves from a horizontal position, about one end as a pivot through one complete revolution, returning to its original position. As the angle thus formed increases, the perpendicular distance from a point on the revolving line to the horizontal line changes according to the following table. Graph (using one horizontal space to represent 15°, and one vertical space to represent 1).

Angle	0°	30°	45°	60°	90°	120°	135°	150°	180°
Distance	0	.5	.71	.87	1	.87	.71	.5	0
Angle		210°	225°	240°	270°	300°	315°	330°	360°
Distance		−.5	−.71	−.87	−.1	−.87	−.71	−.5	0

No.	Sq. Rt.	No.	Sq. Rt.	No.	Sq. Rt.	No.	Sq. Rt.	No.	Sq. Rt.
1	1.000	21	4.583	41	6.403	61	7.810	81	9.000
2	1.414	22	4.690	42	6.481	62	7.874	82	9.055
3	1.732	23	4.796	43	6.557	63	7.937	83	9.110
4	2.000	24	4.899	44	6.633	64	8.000	84	9.165
5	2.236	25	5.000	45	6.708	65	8.062	85	9.220
6	2.449	26	5.099	46	6.782	66	8.124	86	9.274
7	2.646	27	5.196	47	6.856	67	8.185	87	9.327
8	2.828	28	5.292	48	6.928	68	8.246	88	9.381
9	3.000	29	5.385	49	7.000	69	8.307	89	9.434
10	3.162	30	5.477	50	7.071	70	8.367	90	9.487
11	3.317	31	5.568	51	7.141	71	8.426	91	9.539
12	3.464	32	5.657	52	7.211	72	8.485	92	9.592
13	3.606	33	5.745	53	7.280	73	8.544	93	9.644
14	3.742	34	5.831	54	7.349	74	8.602	94	9.695
15	3.873	35	5.916	55	7.416	75	8.660	95	9.747
16	4.000	36	6.000	56	7.483	76	8.718	96	9.798
17	4.123	37	6.083	57	7.550	77	8.775	97	9.849
18	4.243	38	6.164	58	7.616	78	8.832	98	9.899
19	4.359	39	6.245	59	7.681	79	8.888	99	9.950
20	4.472	40	6.325	60	7.746	80	8.944	100	10.000

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